AN INCONSISTENCY IN ARITHMETIC

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During the first part of the 20^{th}. Century, Bertrand Russell developed his so-called Russell's Paradox or Russell's Antinomy. Briefly stated, the paradox concerns itself with a particular aspect of set theory which describes the set of all sets that do not contain themselves.

The set of all sets that do not contain themselves is written in terms of meta-mathematics and formal logic and is defined intentionally as:

$$R = \{S : S \notin S\}$$

The question we now ask is: "Does *R* contain itself?" Let us suppose that *R* does contain itself and we have the hypothesis:

$$R \in R$$

and thus we infer that:

$$R \in \{S : S \notin S\}$$

and therefore we can conclude that:

$$R \notin R$$

which is a contradiction of our original hypothesis.

Now let us suppose that *R* does NOT belong to itself and that we have the hypothesis:

$$R \notin R$$

and thus we infer that:

$$R \notin \{S : \notin S\}$$

and therefore we can conclude that:

$$R \notin R$$

which again is in total contradiction of our original hypothesis.

An appreciation of the implications of the Russell paradox leads us to consider its consequences when applied to other branches of mathematics.

All the previously stated expressions are written in the form of meta-mathematics and this being the case, we can utilise the techniques developed by the great mathematician Kurt Godel which will enable us to express the Russell paradox in an arithmetic format.

It will be recalled that Godel developed a system known as Godel numbering which enables a meta-mathematical expression to be allocated a unique number or integer which can only ever refer to that particular meta-mathematical expression. The core of Godel's system was to allocate an integer to each meta-mathematical symbol employed in the structure of a meta-mathematical expression. Each symbol within the expression would then be allocated a prime number chosen from the whole lexicon of prime numbers and that particular prime number would then be raised to the power of the integer previously allocated to each meta-mathematical symbol as previously described.

It is understood that this Godel numbering system may at first glance appear to be convoluted and over-complicated but it is not, and it can best be illustrated by utilising the expression $\{S: S \notin S\}$ as an example.

Firstly we begin by allocating Godel numbers to each of the symbols in the expression as follows:

$\}$ = 1 and means 'parenthesis'

$\{$ = 2 and means 'parenthesis'

S = 3 and means 'set'

: = 4 and means 'such that'

$\notin$ = 5 and means 'is not a member of' or 'is not an element of'

We then allocate a prime number to each symbol in the expression and then raise that prime number to the power of the Godel number allocated in the list above.

The whole expression can now be expressed in numerical terms as follows;

$$\begin{array}{ccccccccccccccccc} & & \{ & & S & & : & & S & & \notin & & S & & \} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ R & = & 2^2 & \times & 3^3 & \times & 5^4 & \times & 7^3 & \times & 11^5 & \times & 13^3 & \times & 17^1 \\ & = & \multicolumn{13}{l}{1.3926446 \text{ x } 10^{17}} \end{array}$$

Clearly this is a huge number and it is very obvious that some expressions may well be beyond the capacity of many computers to process. Nevertheless, the principle is clear—there can be constructed a unique integer for every conceivable meta-mathematical expression and thus in principle we can treat every meta-mathematical statement in terms of arithmetic. This being the case it is now possible for us to examine the Russell paradox in purely arithmetic terms.

Returning now to the original expression for the paradox, that is to say that the set of all sets that do not contain themselves is defined intentionally as:

$$R = \{S : S \notin S\}$$

As previously described, we have been able to allocate to this expression an integer which for the sake of convenience we will call R' (rather than utilising the whole many-digit expression).

Having now established an arithmetical expression for the value of R', let us consider again the two fundamental questions posed by the paradox. Firstly, we ask—Does the set R' contain itself? Let us suppose that it does contain itself and we have the hypothesis:

$$R' \in R' \qquad \text{(Hypothesis 1)}$$

and so:

$$R' \in \{S\colon S \notin S\}$$

and we conclude that;

$$R' \notin R'$$

or more particularly we can say that since $R' \notin R'$ then the Godel number for $\{S\colon S \notin S\}$ is not contained in the Godel number for $\{S\colon S \notin S\}$. Clearly a contradiction.

Now we suppose that R' does not contain itself and we have the hypothesis:

$$R' \notin R' \qquad \text{(Hypothesis 2)}$$

and so:

$$R' \notin \{S\colon S \notin S\}$$

and we conclude that:

$$R' \in R'$$

Or more particularly we can say that since $R' \in R'$, then the Godel number for $\{S\colon S \notin S\}$ is contained within the Godel number for $\{S\colon S \notin S\}$ which contradicts the supposition that R' does NOT contain itself.

Now, reminding ourselves that the Russell paradox intentionally defines that the set of all sets that do not contain themselves is an equality expressed as:

$$R = \{S\colon S \notin S\}$$

then we can conclude that in respect of both hypotheses described above that we can write $R' \neq R'$ in respect of hypothesis 1 and we can write $R' = R'$ in respect of hypothesis 2.

In other words, we can have a numerical value or integer which either is or is not equal to itself. Furthermore, we can say that an expression does or does not say of itself that it is either true or false and from which we can infer that an inconsistency exists in certain arithmetical expressions.

END

With grateful acknowledgement to the Russell paradox and to Godel's Incompleteness theorem.