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CONSUMPTION, PRODUCTION, INFLATION AND INTEREST RATES
A Synthesis*

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This paper uses discrete-time and continuous-time models to derive equilibrium relations among real and nominal interest rates and the expected growth, variance and covariance parameters of optimally chosen paths for aggregate real consumption and aggregate production. Simple, intuitive and fairly general relations are obtained which apply to most of the models of financial economics of the past 20 years. The single-good analysis generalizes and provides a synthesis of many prior works, whereas the multi-good analysis provides more original results. Consistent business cycle movements are examined for interest rates, inflation and consumption and production aggregates.

1. Introduction

In a single-good, continuous-time model, Cox, Ingersoll and Ross (1985) – hereafter also referred to as CIR – derived a relation of the instantaneous interest rate to the mean and covariance structure of returns in production processes. Rubinstein (1976, 1981) in a discrete-time model, and Garman (1977) and Cox and Ross (1977) derived relations of the parameters of optimal consumption paths to interest rates.[1] This paper provides a synthesis of the relations among interest rates and optimally chosen consumption and production paths in an economy with uncertainty and inflation. The economic analysis of interest rates and optimal production policies significantly extends and generalizes the Cox, Ingersoll and Ross (1985) analysis. Additionally, the paper utilizes consumption aggregation results that are not in those earlier papers. Simple, intuitive and fairly general relations of interest rates to consumption and production aggregates are obtained. Properties of an optimal

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[1] Of course, much economic analysis of consumption, production and interest rates in certainty models precedes those works and certainly precedes this. See Hirshleifer (1970, particularly pp. 116–117) for an excellent discussion of the general equilibrium relationships in a certainty model.

aggregate consumption function are derived and used to explain how the consumption and production results are consistent.

The paper utilizes two standard economic models to examine consumption, production, inflation and interest rates. The principal results can be seen in both models. In section 2, a discrete-time, multi-period state preference model is first used to develop the relations of consumption growth and of production opportunities to the term structure of interest rates in a single-good economy. Section 3 discusses consistent movements in these variables during a business cycle. In section 4, following CIR, the continuous-time model is used to examine the production and interest rates relation in some detail. Section 5 provides the corresponding continuous-time relation of consumption and interest rates, and section 6 provides a synthesis of the consumption and production results in a single-good economy.

Sections 7 and 8 derive nominally riskless and real riskless interest rates in a multi-good economy; they are much more complex than in a single-good economy. These complexities are likely to be economically significant for analyses of nominal interest rates, since it is generally assumed that movements in anticipated inflation are of the same order of magnitude as the movements of the 'real' interest rate. In much of the multi-good analysis, individuals are assumed to have time-additive, but otherwise general preferences for bundles of consumption goods. Individuals' vectors of budget shares are not assumed to be identical, which virtually ensures that they will measure inflation differently. In the multi-good economies examined, Divisia's price indices are used to show four significant points: (1) the positive relation between the interest rate and the expected growth of aggregate real consumption is essentially unchanged from the similar single-good relation; (2) the relevant inflation rate for the 'Fisher' effect is measured by goods' percentage price changes multiplied by their respective aggregate marginal (not average) expenditure shares and summed; (3) the equilibrium nominal interest rate should include a risk premium or discount proportional to the negative of the covariance of inflation with real consumption; and (4) the negative relation of interest rates to the variance of aggregate real consumption does not unambiguously follow from the decreasing absolute risk aversion assumption. However, note that in an economy with Cobb–Douglas preferences, the negative relation of interest rates to consumption uncertainty does hold, as is shown. A variant of the Fisher equation is a special case of the model.

Section 9 concludes the paper with a few comments on the limitations and possible future extensions of the theory.

2. A state preference model of consumption, production and interest rates

The time–state preference model originated with Arrow (1953) and Debreu (1959), and was significantly elucidated by Hirshleifer (1970). Some of the more recent asset pricing papers that used the time–state preference approach are those by Fama (1970), Beja (1971), Rubinstein (1974, 1976a, b, 1981), Kraus and Litzenberger (1975), Hakansson (1977), Banz and Miller (1978), Breeden and Litzenberger (1978), Grauer and Litzenberger (1979), Bhattacharya (1981), and Constantinides (1982). Other significant discrete-time, multi-period valuation models that could easily be rephrased in terms of state–preference are those by Long (1974), Dieffenbach (1975), Brennan (1979), Lucas (1978), and Long and Plosser (1983).

The continuous-time economic model of consumption and portfolio choice was pioneered by Merton (1971, 1973) and extended to a production economy by Cox, Ingersoll and Ross (1985). Other well-known continuous-time asset pricing models include papers by Garman (1977), Cox and Ross (1977), Breeden (1979), and Stulz (1981), as well as the entire literature on option pricing that was begun by Black and Scholes' (1973) seminal work. Since almost all of the general (not arbitrage-based) discrete- and continuous-time asset pricing models assume time-additive utility functions and homogeneous beliefs (the crucial assumptions), the equilibrium interest rate derivations that follow should characterize these models.

In both the time–state preference model and the continuous-time model, the following assumptions are made: (A.1) The economy has a single physical good. (This assumption is relaxed in sections 7 and 8.) Since individuals' preferences are based entirely upon the consumption of this single good, the interest rate analysis of sections 2–6 should be interpreted as applying to 'real' rates. (A.2) In the time–state preference economies examined, there are N risky production processes that have non-increasing returns to scale. In the continuous-time economy of section 4, the stronger assumption of (A.2′) stochastically constant returns to scale is made (as in the CIR paper). Individuals allocate their wealths to production processes and to state-contingent financial claims. (A.3) Individuals have homogeneous probability beliefs for future states of the world, with the time 0 probability for the occurence of state θ at time t defined as $\pi_{t\theta}$. (A.4) Each individual $\{k\}$ maximizes the expected value of a time-additive and state-independent von Neumann–Morgenstern utility function for lifetime consumption, i.e.,

$$\max_{\{c^k_{t\theta}\}} \sum_t \sum_{\theta \in S(t)} \pi_{t\theta} \cdot u^k(c^k_{t\theta}, t), \tag{1}$$

where $c^k_{t\theta}$ is k's consumption of the good at time t if state θ occurs. It is assumed that $u^k(c^k_t, t)$ is monotonically increasing and strictly concave in consumption, displays decreasing absolute risk aversion (which implies that $u^k_{ccc} > 0$), and has marginal utility that approaches infinity as consumption approaches zero. Finally, (A.5) it is assumed that the capital markets are sufficiently complete to permit an unconstrained Pareto-optimal allocation of time–state contingent consumption claims.

With a Pareto-optimal allocation, the shadow price for one unit of the good to be received in time–state $t\theta$ is the same for all individuals; let it be $\phi_{t\theta}$. The standard first-order condition is that $\phi_{t\theta}$ equals the marginal rate of substitution of consumption today for consumption in time–state $t\theta$:

$$\phi_{t\theta} = \frac{\pi_{t\theta} u'^k\left(c^k_{t\theta}, t\right)}{u'^k_0\left(c^k_0, t_0\right)}, \quad \forall k. \tag{2}$$

Note that ranking states at time t in order of their price-to-probability ratios, $\{\phi_{t\theta}/\pi_{t\theta}\}$, gives an exactly inverse ranking of every individual's optimal consumption in the various possible states at time t. Thus, each individual's optimal consumption function may be written in 'reduced form' as a strictly monotonic function of only aggregate consumption, $C_{t\theta}$, and time, $c^k_{t\theta} = c^k(C_{t\theta}, t)$, as shown by Breeden and Litzenberger (1978). Letting $u'(C_{t\theta}, t) = u'^k_t(c^k(C_{t\theta}, t), t)$ for some k, the value of any asset with time–state contingent payoffs $\{X_{t\theta}\}$ equals in equilibrium

$$V_0\{X_{t\theta}\} = \sum_t \sum_{\theta \in S(t)} \phi_{t\theta} X_{t\theta} = \sum_t \frac{\mathrm{E}_0\left[\tilde{X}_t u'(\tilde{C}_t, t)\right]}{u'(C_0, t_0)}. \tag{3}$$

With Pareto-optimal capital markets, this valuation equation holds at every instant, with the relevant probabilities being those conditional upon all information available at the time and state of valuation. Thus, the value of any asset at any time and state may be written in terms of only its payoffs' (conditional) joint probability distributions with aggregate consumption.

From (3), the value at time t of a riskless unit discount bond maturing at T, $B(t, T)$, and the associated continuously-compounded interest rate for the period, $r(t, T)$, are[2]

$$B(t, T) = \mathrm{e}^{-r(t,T)(T-t)} = \frac{\mathrm{E}_t\left[u'(\tilde{C}_T, T)\right]}{u'(C_t, t)}, \qquad \forall t, \quad T > t. \tag{4}$$

Let subscripts of the u' function be partial derivatives, and let $m_{(n)}(t, T)$ be the nth central moment for $\ln C_T$ as seen at time t. For $n = 3$, this gives the skewness of $\ln C_T$, $n = 4$ gives its kurtosis, and so on. Expanding (4) in a

[2] Rubinstein (1976b) derived (4) with the assumption that all individuals have isoelastic utility functions with the same power. This derivation shows that the relation of bond prices to the probability distribution for aggregate consumption does not require such strong preference assumptions.

Taylor series about the current time and the current level of the log of aggregate consumption, $\{t, \ln C_t\}$ gives the following term structure approximation:

$$r(t,T) = [-u'_t/u'] + [-u'_{\ln C}/u']\mu_{\ln C}(t,T)$$
$$- \left[\tfrac{1}{2}(u'_{\ln C,\ln C}/u')\right]\sigma^2_{\ln C}(t,T)$$
$$- \sum_{n=3}^{\infty}(1/n!)\left[u'_{(n)}/u'\right]m_{(n)}(t,T) + \text{higher-order terms}, \tag{5}$$

where

$$\mu_{\ln C}(t,T) = \frac{\mathrm{E}_t(\ln\tilde{C}_T) - \ln C_t}{T-t} \quad \text{and} \quad \sigma^2_{\ln C}(t,T) = \frac{\mathrm{var}_t(\ln\tilde{C}_T)}{T-t}.$$

The bond price equation and the term structure approximation are worthy of further discussion. They say that the entire term structure of interest rates at every point in time may be written in terms of just (1) time, (2) the current level of aggregate consumption, and (3) the probability distributions for aggregate consumption at the maturity dates of the bonds examined. Utility functions are quite general and diverse within the time-additive class, and no assumptions about the stochastic processes for production have been made. Thus, the model applies to most of the general asset pricing models in the finance literature of the 1970s.

Other items that might have been in the bond pricing equation, but are not, include aggregate wealth, the distribution of wealth, the composition of consumption, past consumption levels, and the parameters of the production possibility frontier. Of course, the fact that these variables do not explicitly appear is the result of the endogenous nature of aggregate consumption. Aggregate consumption reflects aggregate wealth and its distribution, as well as production possibilities. Similarly, the probability distribution for future levels of aggregate consumption reflects both initial wealth and the production possibility set.

With the much stronger assumption [see Rubinstein (1974)] that (A.6) individuals' preferences give an aggregate utility function of

$$u(C_t, t) = \mathrm{e}^{-\rho t}C_t^{1-\gamma}, \tag{1'}$$

which has constant relative risk aversion equal to γ, and (A.7) aggregate consumption at time T is lognormally distributed as seen at time t, the term structure equation becomes much simpler:[3,4]

$$r(t,T)=\rho+\gamma\mu_{\ln C}(t,T)-(\gamma^2/2)\sigma^2_{\ln C}(t,T). \tag{6}$$

While the CRRA and lognormal assumptions made for (6) are probably not bad first approximations for preferences and for probability distributions, they are almost surely inconsistent as an exact model of a stochastic term structure.[5] However, section 6 shows that in a continuous-time model, such a relation describes the instantaneous riskless rate for general preferences and general probability distributions (generated by diffusion processes).

The intuitive basis for the positive relation of interest rates to time preference is well-known. The higher the measure of pure time preference [ρ in eq. (6), $-u'_t/u'$ in eq. (5)], the greater the relative preference for goods today. Thus, the higher the rate of time preference, the higher the interest rate must be to induce individuals to defer consumption and buy a bond.

The positive relation of the riskless rate to expected consumption growth is also easily understood. The price of any discount bond is equal to the expected marginal utility of consumption at the maturity date, divided by the marginal utility of consumption today. With the maintained assumptions that each individual has decreasing marginal utility and decreasing absolute risk aversion, the relation of future marginal utility to (uncertain) future consumption can be graphed as in fig. 1. Holding current consumption constant and shifting the probability distribution of future consumption towards higher levels decreases expected future marginal utility, which decreases the bond price and increases the interest rate. Therefore, the riskless rate is positively related to the expected change in the log of consumption, $\mu_{\ln C}$.

[3]To derive (6), substitute marginal utilities for (1′) into (4) to get:

$$B(t,T)=\mathrm{e}^{-r(t,T)(T-t)}=\mathrm{e}^{-\rho(T-t)}\mathrm{E}_t\left[(\tilde{C}_T/C_t)^{-\gamma}\right]. \tag{4′}$$

Next, if $\tilde{C}_T/C_t$ is lognormal with the log's mean equal to $(T-t)\mu_{\ln C}$ and its variance equal to $(T-t)\sigma^2_{\ln C}$, then $(\tilde{C}_T/C_t)^{-\gamma}$ is lognormal with its log's mean of $-(T-t)\gamma\mu_{\ln C}$ and its variance equal to $(T-t)\gamma^2\sigma^2_{\ln C}$. Substituting these into (4′), using the fact that $\mathrm{E}(\mathrm{e}^{\tilde{x}})=\exp[\mu+\sigma^2/2]$ when $\tilde{x}$ is normal, and taking logs of both sides gives (6).

[4]The CRRA–lognormal term structure of (6) was derived independently by Garman (1977, p. 39) and Breeden (1977, ch. 7). This combination of CRRA and lognormal consumption assumptions is presented solely as an example that permits an exact identification of coefficients in the more general term structure of (5). Consumption is clearly an endogenous function of wealth, the production opportunity set, and time, and its probability distribution ideally should be derived from preferences and the joint probability distribution of those more fundamental variables.

[5]An exact continuous-time model with CRRA preferences and consumption endogenously determined to be lognormally distributed implies a constant term structure of interest rates. I thank John Cox and Chi-Fu Huang for pointing this out to me. However, the approximation of (5) is quite general and is consistent with a stochastic term structure, and (6) may be viewed as a mean–variance version of it. Furthermore, section 5's quite general derivation of the instantaneous rate in terms of instantaneous parameters for aggregate consumption yields the same equation as (6).

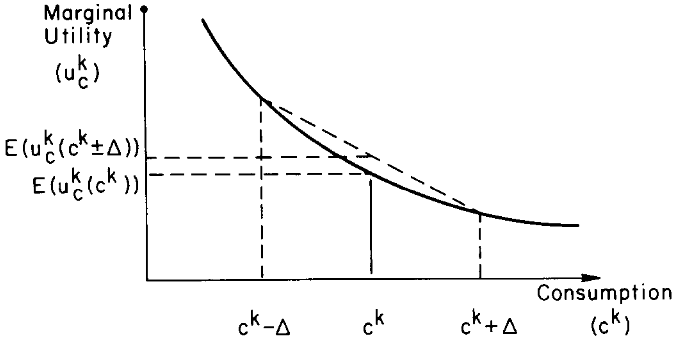

Fig. 1. This figure shows that increases in consumption variance increase expected marginal utility. It compares the expected marginal utility of an individual who consumes c^k for sure to the expected marginal utility of an individual who consumes either $c^k + \Delta$ or $c^k - \Delta$ with equal probability.

To see the negative relation of interest rates to the variance of consumption, consider the effects of a mean-preserving spread of the distribution for consumption at the maturity date (again holding current consumption constant). This is illustrated in fig. 1 by taking probability from the expected future consumption level, $\bar{c}^k$, and splitting it into increased probabilities for levels $\bar{c}^k + \Delta$ and $\bar{c}^k - \Delta$. Due to the decreasing absolute risk aversion assumption, $u_{ccc}^k > 0$ for each individual k, so the increased variance for consumption increases expected marginal utility in the future. This is consistent only with a higher bond price and a lower interest rate. Intuitively, the greater the uncertainty about consumption that will be optimal at time T, the greater the value of the certain payoff provided by a bond maturing at that time, *ceteris paribus*.

Now consider the production side of the economy. For an optimal policy, k's marginal utility for wealth at any time and state equals her marginal utility for consumption in the same state and time, $u_c^k(c^k(Q^k, s, t), t)$, where s is a vector of variables that describe the state of production opportunities and Q^k is k's wealth. Corresponding to the result that each individual's optimal consumption is a monotonic function of only aggregate consumption, we have the following theorem on optimal wealth allocations in a production economy:[6]

Theorem. Optimal Allocations With Production. If (A1) each individual has a time-additive, state-independent utility function for lifetime consumption and (A2) individuals agree upon the (conditional) probabilities of states at every point in time, then any unconstrained Pareto-optimal allocation of resources to production processes and of time–state contingent consumption claims is such

[6] For related theorems, see Constantinides (1982) and Breeden (1984).

that, at each date, all states with the same level of aggregate supply of the good and the same investment opportunities have the same allocation of wealth to individuals in the corresponding competitive economy. With diminishing marginal utility for wealth, this implies that individual k's optimal amount of the good at time t may be written as a strictly positive monotonic function of only the aggregate *amount of wealth at that time, given the state vector for production opportunities and time, i.e.,* $Q_t^k = F^k(Q, s, t)$, *with* $F_Q^k > 0$.

An outline of the proof of this theorem is in appendix 1.

An important implication of this theorem is that neither the past path of production, nor the past path of the state vector for production opportunities should affect an optimal allocation, given the current production opportunity set and the current aggregate supply of the good. This theorem also implies that if the allocation is Pareto-optimal, the distribution of wealth to individuals is not needed as a descriptor of the system at every point in time. Given knowledge of the initial wealth distribution, preferences and probability beliefs, all future optimal distributions of wealth are fully determined by aggregate wealth and the state of production opportunities.

Let individual k's expected utility for lifetime consumption be given by $J^k(Q^k(Q, s, t), s, t)$. From the optimality condition that the marginal utility of wealth equals the marginal utility of consumption, production-oriented valuation equations obtain which are similar to the consumption-oriented equations (3) and (4). Defining $J'(Q, s, t) = J_Q^k(Q^k(Q, s, t), s, t)$ for some k, and substituting the envelope condition into (3) and (4) gives

$$V_0\{X_{t\theta}\} = \sum_t \frac{\mathrm{E}_0[\tilde{X}_t J'(\tilde{Q}_t, \tilde{s}_t, t)]}{J'(Q_0, s_0, t_0)}, \tag{7}$$

$$B(t,T) = \mathrm{e}^{-r(t,T)(T-t)} = \frac{\mathrm{E}_t[J'(\tilde{Q}_T, \tilde{s}_T, T)]}{J'(Q_t, s_t, t)}, \qquad \forall t, \quad T > t. \tag{8}$$

As (8) reflects the sacrifice of $B(t, T)$ at t for a riskless return of unity at T, it arises from equilibrium in the markets for 'riskless intertemporal exchanges'. An equation reflecting an 'intratemporal equilibrium' in the markets for risk and return is obtained as follows. In equilibrium, consider investing at time t one more unit in a risky portfolio or in an active production process that pays $\tilde{x}_T$ at time T per unit invested. That is, $x_{Ts} = \partial X_{Ts}/\partial X_t$, so $\tilde{x}_T$ is the marginal return on investment. In equilibrium, one unit must be the value at t of this, so $1 = \mathrm{E}_t[\tilde{x}_T \tilde{J}_T']/J_t'$, from (7). Combining this with (8), the price of a riskless bond maturing at T may be written as

$$B(t,T) = \mathrm{E}_t[J'(\tilde{Q}_T, \tilde{s}_T, T)]/\mathrm{E}_t[\tilde{x}_T J'(\tilde{Q}_T, \tilde{s}_T, T)]. \tag{9}$$

Thus, this equilibrium condition represents indifference (at the margin) between the returns received at T from additional risky investment and the returns at T from the riskless investment. It is in this sense that (9) represents equilibrium at t in the market for intratemporal risks resolved at T.

In a manner similar to that used for consumption, first-order Taylor series approximations for the term structure can be obtained from (8) and (9). The expansion of (9) gives the following:[7]

$$\begin{aligned} r(t,T) = {} & \mu_x(t,T) - [\partial \ln J'/\partial \ln Q]\sigma_{x,\ln Q}(t,T) \\ & + [\partial \ln J'/\partial s]V_{xs}(t,T) \\ & + \text{terms with higher-order co-moments of } x \text{ with } \{\ln \tilde{Q}, \tilde{s}\}, \end{aligned} \tag{10}$$

where

$$\mu_x(t,T) = \ln[\mathrm{E}_t(\tilde{x}_T)]/(T-t),$$

$$\sigma_{x,\ln Q}(t,T) = \operatorname{cov}(\ln \tilde{x}_T, \ln \tilde{Q}_T)/(T-t),$$

$$V_{xs}(t,T) = \operatorname{cov}(\ln \tilde{x}_T, \tilde{s}_T)/(T-t).$$

This production-oriented equation suggests some interesting possible term structures. Consider that at any point in time there are productive investments that are being made that result in output and profits at various dates in the future. Some investments take one year before production is forthcoming, some take two years, and so on. Some investments can be brought into production more quickly, but at higher cost. For all active processes in which current investment is being made, (10) must hold. Holding constant the covariances of production with wealth and the state variables, the term structure of interest rates is seen to mirror the 'term structure of expected returns on investments'.[8] Since the cost of production is usually assumed to

[7]There are two more obvious Taylor series approximations that are not presented in the paper – an approximation of (8) for the term structure, and of (3) for an arbitrary asset. The approximation of (8) gives a term structure that could be analyzed much like what follows. The approximation of (3) gives a multi-moment consumption-oriented CAPM.

[8]Note that since these are simultaneous relationships, one could just as correctly say that the optimal marginal returns on investments adjust to the term structure of interest rates. However, with the constant returns to scale technology examined later in section 4, the text's statement is more consistent. Note also that Fama and Gibbons (1982) found empirical evidence supporting this theoretical relation of real interest rates to expected returns from production. They found that increases in the *ex ante* real rate were associated with increased subsequent capital investment, presumably induced by increased expected returns on investment. Similarly, they found that real returns on capital were positively correlated with the *ex ante* real rate.

decrease (and profitability increase) as production time increases, the term structure should have an upward slope under normal circumstances. In that case, short-term interest rates would be lower than long-term interest rates in equilibrium, due to the lower returns from their principal competitors – short-term productive investments.[9] However, if (at the optimum) expected profitability is higher for active short-term investments than for active long-term investments, this term structure can also give a downward-sloping term structure. Again, the import of the theory is that the term structure of interest rates mirrors the term structure of (marginal) expected returns on productive investments.

The interpretations of the uncertainty terms in this term structure are similar to those for the instantaneous rate, which will be examined in detail in the next section. However, there is one difference that should be noted. The covariance of the investment's return with aggregate wealth at time T is a portion of the risk adjustment for the investment, as it is in the instantaneous case. The difference is that, in the instantaneous case, the stochastic properties of the aggregate supply of the good are completely determined by the instant's production uncertainty; over discrete intervals, the probability distribution for aggregate wealth reflects consumption withdrawals in the interim, and they are endogenous functions of wealth, the state vector and time throughout the interval. One would still expect that an investment that has a positive correlation with aggregate output at each instant would have a positive correlation with aggregate wealth over a discrete interval, so the basic intuition does not change. But if consumption withdrawals were positively related to aggregate output, the wealth variable in (10) is a smoothed version of the returns to a rolled-over portfolio of investments in the economy's technologies.

3. Fluctuations of interest rates during a business cycle

While it is recognized again that consumption, production and interest rates are all endogenous variables, the term structure equations derived can be used to place restrictions on their equilibrium joint stochastic processes. For exam-

[9] In an insightful paper, Hirshleifer (1971) arrived at similar results, but for different reasons. Hirshleifer examined the term structure effects of the relative illiquidity of long-term point-input point-output production processes. Rolled-over investments in short-term production processes allow intermediate reallocations of goods between investment and consumption ('flexibility'); long-term production processes have little or no such flexibility. Intermediate reallocations may be optimal due to the arrival of new information about production possibilities or the distribution of future endowments. Thus, if the probability distributions of long-term and short-term investment returns were identical, individuals would prefer short-term investments. Hirshleifer shows that, in equilibrium, marginal investments in long-term activities must provide higher expected returns than provided on similar short-term activities, which is consistent with an upward-sloping term structure. With decreasing returns to scale, this could be achieved by relatively larger investments in short-term processes (giving them lower marginal products) than in long-term processes. Hirshleifer's model is compatible with this model, so his results must also occur here, given his assumptions.

ple, if one has information that real interest rates are higher now than in the past, a rational prediction (given the assumptions of the model) is that real consumption growth will be higher in the future than in the past, or that it will be less variable. It is in this spirit that the following discusses consistent movements in interest rates over a business cycle. Implicit in this discussion is the assumption that economists and individuals do have changing predictions about real consumption growth, and that those are rationally coordinated with their portfolio decisions.

The real term structure given by the CRRA–lognormal model or by the more general term structure approximation, (6) and (5) respectively, may have a variety of interesting shapes, depending upon the expected growth rate of aggregate consumption, as well as the uncertainty of that growth. In particular, the CRRA–lognormal term structure will be flat and will remain flat over time if the aggregate consumption process is a geometric Brownian motion, even if aggregate consumption is wildly variable. Actually, with CRRA utility, this result does not require the assumption of lognormality, as can be seen by examining the pricing equation for bonds for the special case where u' is a pure power of aggregate consumption. All that is required is that the probability distribution of $\tilde{C}_{t+\Delta}/C_t$ not change over time.

From (6), the T-period real riskless rate will be positively and linearly related to the T-period expected growth rate of aggregate consumption, whereas it will be negatively and linearly related to the T-period average variance rate for real consumption. For a maturing economy with the expectation of a gradually declining rate of growth of real aggregate consumption, the real term structure should tend to be downward-sloping, *ceteris paribus*. However, if the average variance rate of real aggregate consumption, $\sigma_C^2(t, T)$, were a decreasing function of $T-t$, the time to maturity of the associated discount bond, then the term structure would tend to be upward-sloping, *ceteris paribus*. There is some indirect evidence that, *ex post*, the one-year variance rate of aggregate real consumption has declined in the United States since about the year 1900, while average per capita real growth has not changed dramatically over the past 100 years (splitting the 100 years into two 50-year subperiods).[10] If this trend were expected, *ex ante*, then the term structure at that time should have been rising. If this trend were expected to continue, then the term structure should presently tend to be upward-sloping.

Even if one-year variance rates for aggregate consumption are constant over time, it is possible that the (unconditional) multi-period average variance rates in eqs. (5) and (6), $\operatorname{var}(\ln \tilde{C}_t)/(T-t)$, are decreasing in time to maturity, $T-t$. For example, this would occur if one-year (conditional) expected growth rates for aggregate consumption increase from one period to the next when current

[10] See Roberts (1977) for the evidence on declining variability. The evidence on the average per-capita real growth rate was provided by my research assistant, Ehud Ronn, using Kuznets' data for the early years.

consumption unexpectedly declines.[11] Intuitively stated, this is an economy that has negative autocorrelation in one-year real consumption growth rates, but stable one-year variances. Such an economy should tend to have an upward-sloping term structure.

If both the T-period mean growth rate and the T-period average variance rate decline with increasing time to maturity, T, at a given time, then the real term structure may either be rising or falling or be humped, depending upon the parameter values in (6). In that case, the mean effect offsets the variance effect on the term structure. Thus, the term structure equation given is quite flexible in terms of the shapes that may obtain; however, it is not so flexible as to be entirely useless. It predicts the signs of the partial effects of two parameters of aggregate consumption's probability distribution at any future date, one of which is the subject of numerous economists' forecasts (the mean), and the other of which (the variance) may be amenable to estimation.

Holding the maturity structure of average variance rates constant, the cyclical behavior of real rates may be examined by considering cyclical changes in expected economic growth rates. If the expectation is that economic growth will be rapid for a couple of years and then decline, then real interest rates should be 'high' for short-maturity discount bonds and relatively 'low' for long-term bonds. Thus, if the economy is thought to be entering a short-term rapid growth phase (coming out of a recession), real short-term interest rates should be high and the real term structure downward-sloping (or not rising as much as usual). Conversely, when the economy is believed to be entering a period of decline or of very slow growth relative to its long-term expected growth, the real term structure should tend to be rising.[12]

[11] This scenario is plausible for an economy with cyclical fluctuations about a long-term stationary trend. However, Mishkin (1981) found the *ex post* real rate to be unrelated to the gap between potential GNP and actual GNP; a positive relation would be expected with the long-term stationary economy posited. Additionally, Nelson and Plosser (1982) could not find evidence of a tendency for real GNP per capita to return to a deterministic path. Their conclusion is that (from their Abstract) 'macroeconomic models that focus on monetary disturbances as a source of purely transitory fluctuations may never be successful in explaining a large fraction of output variation, and that stochastic variation due to real factors is an essential element of any model of macroeconomic fluctuations'. Thus, Nelson and Plosser's evidence is against this particular hypothetical case. However, since the models derived here are much more general than this one example, their work does not contradict the general relations derived here of real consumption growth and real returns on investment to real interest rates.

[12] Note that movements in the term structure for inflation could reasonably offset these movements in the real term structure, resulting in shifts in the nominal term structure that are opposite to the cyclical predictions of this model. Consider an economy with negatively correlated growth rates in consumption. Furthermore, assume that at times when consumption has grown rapidly, the expected inflation rate is quite high. At a time when expected real short-term consumption growth is quite low relative to anticipated long-term consumption growth, short-term real interest rates should be low, and the real term structure rising. However, at that time, inflation might be very high and be expected to fall with the decreasing growth of the economy. The downward-sloping inflation structure, combined with the upward-sloping real term structure can give a nominal term structure that is either rising, falling or humped. Empirical research by

The term structure equations indicate that the hypothesis that the short-term real rate of interest is constant is rather implausible, as the expected real growth rate of aggregate consumption for small T may fluctuate considerably over a business cycle. Certainly the average of economists' forecasts of real consumption growth varies considerably over time, as does its dispersion. Furthermore, implied standard deviations of stocks' returns (from option prices) vary considerably through time, which at least suggests changes in the general level of economic uncertainty. The hypothesis that the long-term real rate of interest is constant (or at most a function of time) is more plausible, as long-term expected real growth rates and variance rates may be much more stable.

4. A continuous-time model of production and interest rates

The continuous-time model with constant-returns-to-scale production is essentially that of Cox, Ingersoll and Ross (1985), so its principal features will only be outlined. At time t, individual k's wealth is Q^k units of the good. From that wealth, at each instant, k chooses: (1) a consumption rate, c^k, (2) an optimal vector of investments in N risky production processes, $\boldsymbol{q}^k$, (3) an amount to be lent risklessly (borrowed if negative), q_0^k, and (4) a portfolio of investments in risky financial assets, $\boldsymbol{w}^k$. Individual k's resource constraint is $\sum_i q_i^k + q_0^k + \sum_j w_j^k = Q^k$. Individuals again are assumed to have time-additive preferences, maximizing an integral version of (1). However, the slightly more specialized assumption that $u^k(c^k, t) = \mathrm{e}^{-\rho t} U^k(c^k)$ is used in this section.

It is assumed that holdings of financial assets may be long or short for anyone, but that aggregate supplies of financial assets are all zeroes (since production is done only by individuals). With constant returns to scale and the unlimited borrowing assumption, individuals can do what firms could do. Thus, the financial assets are 'side bets' in this economy. Market clearing conditions at each point in time are $\sum_k \boldsymbol{w}^k = \boldsymbol{0}$ and $\sum_k Q^k = Q$. The vector of aggregate amounts invested in the various production processes is $\boldsymbol{q}^M = \sum_k \boldsymbol{q}^k$.

Each individual has the same production opportunity set, with the output from production process i being governed by a stochastic differential equation of the following type:

$$\mathrm{d}q_i^k = q_i^k \mu_{qi}(\boldsymbol{s}, t)\,\mathrm{d}t + q_i^k \sigma_{qi}(\boldsymbol{s}, t)\,\mathrm{d}z_{qi}, \tag{11}$$

where $\boldsymbol{s}$ is an $S \times 1$ vector of state variables that follow a vector Markov process with drift and diffusion parameters $\boldsymbol{\mu}_s(\boldsymbol{s}, t)$ and $\boldsymbol{\sigma}_s(\boldsymbol{s}, t)$, respectively.

Mishkin (1981) shows a strong negative correlation of the real rate with expected inflation, which results in low real rates typically when nominal rates are 'high'. Quoting Mishkin (1981, p. 173): 'When nominal rates are high, it is more likely that we are in a period of 'easy money' with low real rates than the contrary as has frequently been assumed.'

From (11), both uncertain production rates and random technological change are modeled, the former through the $\mathrm{d}z_q$ term, and the latter through the impact of stochastic fluctuations in the state variables on the means and uncertainties of the various production processes. It will be argued that the role of the financial assets in this model is to allocate the risks of changes in production technologies.

Financial asset j has a price P_j and pays no dividends. Financial assets' prices are endogenous functions of wealth, the state vector and time, moving stochastically through time as Ito processes. [See Huang (1983) for a rigorous model of information and asset prices in a continuous-time economy.] The $A \times 1$ vector of instantaneous expected returns on risky assets is $\boldsymbol{\mu}_a(Q, \boldsymbol{s}, t)$, with incremental covariance matrix $\mathbf{V}_{aa}$, covariances with production rates given by the $A \times N$ matrix $\mathbf{V}_{aq}$, and covariances with the state vector in the $A \times S$ matrix, $\mathbf{V}_{as}$. All of these covariance matrices may depend upon the state vector.

First-order conditions for the optimal production inputs and the optimal asset portfolio imply [see CIR (1985, eq. 10) and Huang (1983)]:

$$\begin{bmatrix} \boldsymbol{q}^k \\ \boldsymbol{w}^k \end{bmatrix} = T^k \begin{bmatrix} \mathbf{V}_{qq} & \mathbf{V}_{qa} \\ \mathbf{V}_{aq} & \mathbf{V}_{aa} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\mu}_q - \boldsymbol{r} \\ \boldsymbol{\mu}_a - \boldsymbol{r} \end{bmatrix} + \begin{bmatrix} \mathbf{V}_{qq} & \mathbf{V}_{qa} \\ \mathbf{V}_{aq} & \mathbf{V}_{aa} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{V}_{qs} \\ \mathbf{V}_{as} \end{bmatrix} \boldsymbol{H}_s^k \tag{12}$$

$$= T^k \mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{r}) + \mathbf{V}^{-1}\mathbf{V}_{qa,s}\boldsymbol{H}_s^k, \tag{12'}$$

where $\mathbf{V}$ and $\boldsymbol{\mu}$ are defined as the covariance matrix and expected return vector for the augmented vector of physical productivities and returns on financial assets. The variables T^k and $\boldsymbol{H}_s^k$ are absolute risk tolerance and Merton's (1973) 'hedging' demands, $T^k = -J_Q^k/J_{QQ}^k$ and $\boldsymbol{H}_s^k = -\boldsymbol{J}_{Qs}^k/J_{QQ}^k$. Risk tolerance based upon the direct utility function is defined as $T_c^k = -u_c^k/u_{cc}^k$. Relative risk tolerances are written with the same notation, but with asterisks attached.

Note that individuals differ in their production and investment policies only as they differ in risk tolerance and in hedging preferences. This is due to the combination of assumptions that all have the same production and investment opportunities and that those opportunities all exhibit stochastically constant returns to scale. Furthermore, since this model is formally identical to that of Merton (1973) and Breeden (1979), the expected excess return on any risk production process or asset is given by a 'multibeta' CAPM (with betas measured relative to the market and to the S state variables) and also by a consumption-oriented CAPM. However, since the pricing of risky assets is not the focus of this paper, these results will not be further discussed.

Aggregate investments in the various production processes and financial assets are given by summing individuals' investments, given in (12). Noting that aggregate financial investments are all zeroes, this gives

$$\begin{bmatrix} \boldsymbol{q}^M \\ \boldsymbol{0} \end{bmatrix} = T^M \mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{r}) + \mathbf{V}^{-1}\mathbf{V}_{qa,s}\boldsymbol{H}_s^M, \tag{13}$$

where

$$T^M = \sum_k T^k \quad \text{and} \quad \boldsymbol{H}_s^M = \sum_k \boldsymbol{H}_s^k.$$

Let $\boldsymbol{q}_M^* = \boldsymbol{q}^M/Q$ be the aggregate fractions invested in the production processes, and let $T^{*M} = T^M/Q$ be an aggregate measure of relative risk tolerance. Multiplying (13) by $(\boldsymbol{q}^{M\prime}\ \boldsymbol{0})\mathbf{V}/Q^2$ gives an important relation:

$$\begin{aligned} \boldsymbol{q}_M^{*\prime}\mathbf{V}_{qq}\boldsymbol{q}_M^* &= T^{*M}\boldsymbol{q}_M^{*\prime}(\boldsymbol{\mu}_q - \boldsymbol{r}) + \boldsymbol{q}_M^*\mathbf{V}_{qs}\boldsymbol{H}_s^M/Q \\ &= \sigma_Q^2 = T^{*M}(\mu_Q - r) + \boldsymbol{V}_{Qs}\boldsymbol{H}_s^M/Q, \end{aligned} \tag{14}$$

where σ_Q^2 is the variance of optimal aggregate production (as a fraction of the amount invested), μ_Q is the expected return on the market portfolio of productive investments, and V_{Qs} is the vector of covariances of aggregate production with the state variables.

Rearranging terms in (14) gives the instantaneous riskless rate in terms of the expected growth rate, the variance rate, and the covariances with state variables for aggregate production:

$$r = \mu_Q - \left(1/T^{*M}\right)\sigma_Q^2 + \boldsymbol{V}_{Qs}\left(\boldsymbol{H}_s^M/T^M\right). \tag{15}$$

This was derived by CIR (1985, eq. 14) for an economy with identical individuals. This equation is also the same as the mean–variance part of the term structure approximation in the state preference model's eq. (10).

Before going into a fairly detailed analysis of this relation (and a similar consumption relation), a couple of significant points should be emphasized. First, the riskless rate is positively related to expected aggregate productivity and is negatively related to the variance of productivity, *ceteris paribus*. Second, a truly dynamic analysis is consistent with (15), since all of the terms in it are, in general, stochastic. The model used in the derivation assumed that means, variances and covariances of production returns with each other and with the state vector are functions of time and a stochastic state vector. Of course, the risk aversion and hedging parameters are derived from individuals' indirect utility functions, so they also are stochastic unless stronger preference assumptions are made. For example, logarithmic utility functions for consumption imply that $\boldsymbol{H}_s^M = \boldsymbol{0}$ and that $T^{*M} = 1$, as noted by Merton (1973).

To further examine the relation of the interest rate to production technology, the production plans of different individuals, as well as the 'hedging' term, $\boldsymbol{H}_s^M$, must be analyzed. Let $\boldsymbol{\varepsilon}_s^k$ be individual k's vector of compensating variations in wealth as a percentage of wealth that are required to offset changes in the state variables and keep expected lifetime utility constant. That is, $\boldsymbol{\varepsilon}_s^k = -\boldsymbol{J}_s^k/(J_Q^k Q^k)$. If, for example, there is a state variable s_j that represents technological development by having high values when expected productivity is high, and low values when productivity is low, the percentage compensating variation in wealth for an increase in s_j is negative, i.e., $\varepsilon_{sj} < 0$. For the remainder of this section, assume that (A.8) these percentage compensating variations are invariant with respect to individual k's wealth. Given this assumption, Breeden (1984) has shown that $\boldsymbol{H}_s^k$ and $\boldsymbol{\varepsilon}_s^k$ are related as follows:

$$\boldsymbol{H}_s^k = Q^k[1 - T^{*k}]\boldsymbol{\varepsilon}_s^k. \tag{16}$$

Substituting (16) into (15), the riskless rate may be written as

$$r = \mu_Q - (Q/T^M)\sigma_Q^2 + (1/T^M)\mathbf{V}_{Qs}\left[\sum_k (1 - T^{*k})\boldsymbol{\varepsilon}_s^k Q^k/Q\right], \tag{17}$$

and k's optimal allocation of investment to production processes is [from (12)]

$$\begin{bmatrix} \boldsymbol{q}^M \\ \boldsymbol{0} \end{bmatrix} = T^M \mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{r}) + \mathbf{V}^{-1}\mathbf{V}_{qa,s}\boldsymbol{\varepsilon}_s^k(1 - T^{*k})Q^k. \tag{18}$$

From (18), if all individuals have logarithmic utility functions, then $T^{*k} = 1$ and all individuals invest the same wealth fractions in the various production processes. No trading in risky financial assets takes place in this case. In this case, the riskless rate is equal to the expected return on optimal risky investment, less its variance.[13]

Since ε_{sj}^k is k's percentage compensating variation in wealth for an increase in state variable s_j, $\boldsymbol{\varepsilon}_s^{k\prime}(\mathrm{d}\boldsymbol{s})$ is the net percentage compensating variation in wealth for the random fluctuations in the entire state vector. Given this, $\mathbf{V}_{qa,Ik} = -\mathbf{V}_{qa,s}\boldsymbol{\varepsilon}_s^k$ is the vector of covariances of the various activities' outputs and financial assets' returns with the net change in the value of investment technology to k.

The optimal production plan for k can be interpreted with the same insights as in the multi-period portfolio theory of Merton (1973) and Breeden (1984). The first term in (18) is the locally mean–variance efficient combination of productive investments and financial assets, and the second term adjusts the

[13] Both of these logarithmic utility results were obtained by Kraus and Litzenberger (1975) and Rubinstein (1976a) in exchange economies, and by CIR in the production model.

production plan to either hedge or 'reverse hedge' against technological changes. If k's relative risk aversion exceeds one, then $1 - T^{*k} > 0$ and k will tend to adjust his production plan to hedge against unfavorable technological changes. If k's relative risk aversion is less than one, then k will reverse hedge by allocating more to production processes whose outputs are positively correlated with changes in production technology.[14] Thus, those who are relatively risk-tolerant will tend to invest more in production processes that have positively autocorrelated returns than in processes that have mean-reverting cash flows. Those who are very risk-averse will tend to invest more in mean-reverting processes and less in positively autocorrelated processes.

The equilibrium interest rate is also affected by the nature of production autocorrelations in a way that depends upon individuals' risk aversion functions. If the economy is populated by individuals who exhibit normal hedging behavior (which seems most reasonable), then $1 - T^{*k} > 0$ in (18) and the riskless rate will be negatively related to the degree of autocorrelation of output, holding the mean and variance of the aggregate production rate constant. Intuitively, the high-risk aspect of production processes with positive autocorrelation would make individuals reluctant to invest in them; individuals would be more willing to lend risklessly to others who are more risk-tolerant, thereby lowering the riskless interest rate. If production processes have negative autocorrelation, their multi-period returns are more stable than with no autocorrelation. Thus, relatively risk-averse individuals would wish to lend less and invest more in negatively autocorrelated processes, resulting in a higher equilibrium riskless rate.

With an economy of individuals who are more risk-tolerant than the logarithm, the effects of autocorrelation on the riskless rate are correctly predicted by focusing upon the effects of production autocorrelations on mean multi-period returns (rather than upon the variance). Positive production autocorrelations lead to higher multi-period returns on average than those with no autocorrelation. Individuals who are relatively risk-tolerant will attempt to borrow and invest more in such processes, which results in a higher riskless rate than without autocorrelation. Thus, an economy that is not very risk-averse has a riskless rate that is positively related to the degree of autocorrelation in production processes.

As noted, the allocation of inputs to production processes will, in general, be different for each individual. However, if there exist financial assets that perfectly hedge against technological changes, then separation of production mixes from preferences occurs. If the first financial asset were perfectly correlated with the first state variable, then a multiple regression of the first

[14] Breeden (1984) showed that this reverse hedging policy increases the mean lifetime consumption stream. Since relative risk tolerance is a marginal rate of substitution of mean for variance, it is not too surprising that some individuals will choose high mean and high variance, while others will choose low mean and low variance, depending upon their mrs functions.

state variable on all productivities and all financial assets would give zero investments in all of them, except for the first financial asset (which is the perfect hedge). Similarly, assuming that there are exactly S financial assets, each of which is perfectly correlated with a state variable, implies that the matrix of hedge portfolios simplifies to[15]

$$\mathbf{V}^{-1}\mathbf{V}_{qa,s} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I} \end{bmatrix}.$$

Given this and (12′), with these financial assets as hedges, all individuals have the same mix of production inputs, regardless of preferences and covariances of productivities with the state variables.

There are a couple of points to note about this result. One is that the risk exposure of an individual to technological changes can, with these assumptions, be perfectly controlled by investments solely in financial assets. Still, the aggregate Pareto-optimal investments in production processes should reflect their covariances with technological change and individuals' preferences regarding those technological changes. They do, but now they are reflected in all individuals' production plans in the same way through the $\mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{r})$ term in (12). This is true since $\mathbf{V}$ includes the covariances of productivities with assets' returns, which are assumed to reflect technological changes. The expected excess returns on assets are the market's equilibrium prices for the risks of technological change (as given by a consumption-oriented CAPM). If financial assets do not perfectly reflect those technological changes, then the final $\boldsymbol{H}_s^k$ term in (12) does affect production decisions, as those decisions may then help achieve an optimal exposure to technological risks in ways not possible with financial assets alone.

5. Consumption and interest rates in the continuous-time model

Cox, Ingersoll and Ross (1985, theorem 1) have shown that each individual optimally sets the negative of the expected rate of change in the marginal utility of wealth equal to the instantaneous riskless rate. Letting $\mu_{J'}^k$ and $\mu_{u'}^k$ be k's drift parameters for the marginal utility of wealth and consumption, this implies (along with the envelope condition)

$$r = -\mu_{J'}^k / J'^k = -\mu_{u'}^k / u'^k, \quad \forall k. \tag{19}$$

This result is a limiting case, $T \to t$, of the state preference valuation [eq. (4)] for an instantaneous-maturity bond, since that formula holds for each individ-

[15] In this case, the allocation is Pareto-optimal [Breeden (1984)] and there is local unanimity among investors for production plans. Production and consumption separation occurs as in Hirshleifer (1970, ch. 3).

ual. Intuitively, the higher the riskless rate, the more individuals should defer consumption; this results in a higher optimal growth rate for consumption and a larger expected decline in marginal utility.

Using Ito's Lemma, the marginal utility of consumption for k, $u_c^k(c^k, t)$, has a drift that may be found from k's instantaneous expected consumption change, μ_c^k, and its variance, σ_{ck}^2,

$$\mu_{u'}^k = u_{ct}^k + u_{cc}^k \mu_c^k + \tfrac{1}{2} u_{ccc}^k \sigma_{ck}^2. \tag{20}$$

Substituting (20) into (19), and using the fact that $-u_{ct}^k / u_c^k = \rho^k$ with the time-separable preferences assumed, gives

$$r = \rho^k + \left[-u_{cc}^k / u_c^k\right] \mu_c^k - \tfrac{1}{2}\left[u_{ccc}^k / u_c^k\right] \sigma_{ck}^2, \quad \forall k. \tag{21}$$

Note that this corresponds precisely to the first-order terms in the term structure approximation derived in a discrete-time, state preference model in section 2, eq. (5). The riskless rate is positively related to time preference and to the expected growth rate for consumption and is negatively related to the variance of that growth. Equivalently, as Cox and Ross (1977) and Garman (1977) pointed out, each individual adjusts her expected instantaneous growth rate of consumption to correspond to the difference between the riskless rate and her pure rate of time preference, holding the variance of consumption constant. With the diffusion model's assumptions, consumption is locally normal and, thus, higher-order moments of consumption do not appear in the local return relations. Note that the preferences for which this holds are quite general within the time-additive class.

To derive an aggregate version of this relation in continuous time, let us make use of the assumption that the allocation of time–state consumption claims is Pareto-optimal (A.5); were it not, there would be incentives for individuals to create new claims so as to improve the allocation. As noted in section 2, with an optimal allocation, each individual's optimal consumption rate may be written in 'reduced form' as a function of aggregate consumption and time, i.e., $c^k(Q^k, s, t) = c^k(C, t)$, $\forall k$. Given this, and again defining $u'(C, t) = u_c'^k(c^k(C, t), t)$ for some k, the riskless rate may be written in terms of the expected growth rate and variance rate of aggregate consumption:

$$r = [-u_t'/u'] + [-u_C'/u']\mu_C - \tfrac{1}{2}[u_{CC}'/u']\sigma_C^2. \tag{22}$$

Again, this is an instantaneous version of the general term structure relation to aggregate consumption's distribution, eq. (5). The terms of this were discussed in section 2. However, it is important to note that (22) is an exact relation of the instantaneous riskless interest rate to consumption's growth parameters, and it holds for an economy of individuals with general and diverse time-additive preferences. The drawback of this derivation, relative to the term structure derived in the state preference model, is that this continuous-time version does not (by itself) say anything about interest rates for finite-maturity bonds, whereas the state preference model applies to the entire term structure.

6. Consumption, production, and interest rates: A synthesis

Given a relation of optimal aggregate consumption and interest rates from sections 2 and 5, and a relation of optimal aggregate production and interest rates from sections 2 and 4, a synthesis will be presented. The synthesis is presented for the continuous-time model, since the relations derived in that model are exact equations that can be rigorously analyzed without Taylor series approximations. However, since it was shown that the discrete- and continuous-time models give results that are closely related (as they should be), a similar discrete-time synthesis could be presented.

To simplify the synthesis without losing the main points of economic interest, the preference assumption (A.6) is made, which is that individuals' utility functions aggregate to a power utility function that has pure time preference of ρ and relative risk aversion of γ. No assumption is made about the probability distribution for aggregate consumption, except that it follows an Ito process. With the preference assumption, the principal continuous-time equations for the interest rate in terms of production and consumption parameters, eqs. (17) and (22), are

$$r=\mu_Q-\gamma\sigma_Q^2+V_{\ln Q,s}\left[\sum_k(\gamma-1)\varepsilon_s^k Q^k/Q\right], \tag{17$'$}$$

$$r=\rho+\gamma\mu_{\ln C}-\left(\gamma^2/2\right)\sigma_{\ln C}^2. \tag{22$'$}$$

The riskless rate has been related to expected growth and to the variance of both production and consumption rates. For a minimum level of understanding, a shift in expected productivities should be traced through the optimal consumption function to ascertain that these shifts affect the consumption and production sides of the interest rate equations by the same amounts. A comparison of the coefficients for mean consumption growth and for mean production growth provides little comfort, since the coefficient of the expected return from production in (17′) is one, whereas the coefficient of expected consumption growth in (22′) is a measure of relative risk aversion, which may be much different from one. Similarly, changes in uncertainty in the economy do not obviously affect the variance component of the consumption relation by the same amount as in the production relation. The variance of consumption has a coefficient of $(-\gamma^2/2)$, whereas the production variance rate has a coefficient that is a measure of relative risk aversion, and production's covariances with state variables have coefficients related to hedging preferences.

Thus, simple statements that a 1% increase in the expected return from production (or its variance) results in a 1% increase in expected consumption growth (or its variance) cannot, in general, be made consistent with the interest relations derived. The exception to this statement is the case of logarithmic utility [examined by Kraus and Litzenberger (1975), Rubinstein (1976a), and CIR (1985)], for which coefficients of expected production, production variance, consumption growth and consumption variance are all ones, and the production covariances have zeroes as coefficients in (22′). Since the assumption of logarithmic utility for all individuals is very restrictive, the simple 1–1 intuition must be generalized if we are to provide a true synthesis of the consumption and production relations.

Again, let the state vector $\boldsymbol{s}$ positively reflect technological change through the productivity functions $\boldsymbol{\mu}_q(\boldsymbol{s}, t)$. Consider an increase in the productivities of a number of production processes, i.e., $\mathrm{d}\boldsymbol{s} > \boldsymbol{0}$. For discussion, the increase in aggregate productivity, μ_Q, is assumed to be 1%, the variance of aggregate productivity, σ_Q^2, is assumed to remain constant, and the covariances of aggregate production with the state variables are all assumed not to change. Thus, from (17′), the interest rate rises by 1% with the 1% rise in the expected return in production. If the variance of the rate of growth of consumption does not change, the 1% increase in the interest rate must result in an increase in the expected growth rate of consumption equal to the 1%, divided by the aggregate relative risk aversion of the economy, γ. For example, if relative risk aversion (RRA) equals 2, then the change in expected consumption growth must be 0.5%, whereas if RRA is 0.5, then expected consumption growth must increase by 2% for the 1% productivity increase.

The question now is: How may it be shown that the response of the optimal expected growth rate of consumption to the hypothesized 1% change in expected productivity is inversely related to relative risk aversion? With a Pareto-optimal allocation in this economy, aggregate consumption may be written as a function of aggregate supplies, the state vector, and time, i.e., $C(t) = C(Q, \boldsymbol{s}, t)$. Applying Ito's Lemma to this consumption function gives the expected growth rate and variance rate for aggregate consumption in terms of parameters for production and the state variables:

$$\mu_C = C_Q\left[\mu_Q Q - C\right] + \boldsymbol{C}_s \boldsymbol{\mu}_s + C_t + \tfrac{1}{2}\begin{bmatrix} C_{QQ} & \boldsymbol{C}_{Qs} \\ \boldsymbol{C}_{sQ} & \boldsymbol{C}_{ss} \end{bmatrix} \square \begin{bmatrix} Q^2\sigma_Q^2 & \boldsymbol{V}_{Qs} \\ \boldsymbol{V}_{sQ} & \boldsymbol{V}_{ss} \end{bmatrix},$$

$$\sigma_C^2 = \left(C_Q\ \boldsymbol{C}_s\right)\begin{bmatrix} \sigma_Q^2 Q^2 & \boldsymbol{V}_{Qs} \\ \boldsymbol{V}_{sQ} & \boldsymbol{V}_{ss} \end{bmatrix}\begin{bmatrix} C_Q \\ \boldsymbol{C}_s \end{bmatrix}. \tag{23}$$

From (23), the principal effects of the productivity improvement on expected consumption growth are two: first, the change in productivity per unit of investment, μ_Q, and second, the change in the aggregate consumption rate that is due to the change in the state vector. Holding current supplies and current consumption constant, higher expected productivity implies higher future consumption and, therefore, a higher growth rate of consumption. Holding current supplies and expected productivity constant, as the current consumption rate is increased, productive investment is decreased and, hence, future consumption and the consumption growth rate is decreased. Next we show that the higher risk aversion is, the greater the increase in current consumption is with the hypothesized increase in productivity. By showing that current consumption responds more positively to productivity improvements the higher is risk aversion, we will have shown that the optimal expected consumption growth rate change is negatively related to risk aversion, as was to be shown.

With a Pareto-optimal allocation, the envelope condition for individual k may be written in functional form as

$$u_c^k\left(c^k(C(Q,s,t),t),t\right) = J_Q^k\left(Q^k(Q,s,t),s,t\right), \tag{24}$$

as was discussed. Implicitly differentiating (24) with respect to $\boldsymbol{s}$ gives

$$u_{cc}^k[\partial c^k/\partial C]\boldsymbol{C}_s = J_{QQ}^k[\partial Q^k/\partial \boldsymbol{s}] + \boldsymbol{J}_{Qs}^k, \quad \forall k. \tag{25}$$

Dividing (25) by (24), noting that Breeden (1979) has shown that

$$\partial c^k/\partial C = [-u_c^k/u_{cc}^k]/T_c^M = T_c^k/T_c^M,$$

and then multiplying by $T^k = -J_Q^k/J_{QQ}^k$ gives

$$[-\boldsymbol{C}_s/T_c^M]T^k = (\partial Q^k/\partial \boldsymbol{s}) + \boldsymbol{H}_s^k, \quad \forall k. \tag{26}$$

Summing this across individuals and noting that $\sum_k(\partial Q^k/\partial \boldsymbol{s}) = \boldsymbol{0}$ gives

$$\boldsymbol{C}_s = -\boldsymbol{H}_s^M[T_c^M/T^M]. \tag{27}$$

Combining (27) and (16) gives

$$\boldsymbol{C}_s = [T_c^M/T^M]\sum_k[Q^k(\gamma-1)(-\boldsymbol{\varepsilon}_s^k)/\gamma]. \tag{28}$$

Remembering that $\{T_c^M, T^M, \gamma, -\boldsymbol{\varepsilon}_s^k\}$ are all positive for the economy, it is seen from (28) that aggregate consumption responds positively to productivity increases ($\boldsymbol{C}_s > \boldsymbol{0}$) if relative risk aversion is greater than one. If individuals are more risk-tolerant than the log utility case, then aggregate consumption decreases as productivity improves; this leads to higher expected future consumption and, thus, to a higher expected growth rate for consumption than in the more risk-averse case. Thus, the fact that the coefficient of the expected consumption growth rate in (22′) is relative risk aversion, whereas the coefficient of expected productivity is unity, is due to the dependence of current consumption's response upon aggregate relative risk aversion.

The optimality of the relation of RRA to consumption's response to productivity changes can be seen from a multi-period mean–variance perspective, much like the production analysis of section 4. By having a policy of reducing consumption and increasing investment when production opportunities improve, one produces a consumption stream with a higher lifetime mean and a higher lifetime variance than would be generated by the opposite policy. Thus, since RRA just describes investors' marginal rates of substitution of mean for variance, it is not surprising that relatively risk-tolerant investors would follow that policy. In contrast, relatively risk-averse investors would increase consumption as production opportunities improve, thereby generating a low mean–low variance lifetime consumption path. As the analysis shows, the equilibrium interest rate response to productivity changes thereby depends upon the level of risk aversion in the economy.

7. A multi-good model of interest rates, expected real growth and inflation

In the multi-good, continuous-time economy, let $\boldsymbol{c}_t^k$ be k's vector of consumption rates of the various goods at time t. Define the indirect utility function, U_t^k, for nominal expenditure at time t by individual k, e_t^k, as a function of the spot commodity price vector, $\boldsymbol{P}$, in the usual way:[16]

$$U_t^k\left(e_t^k, \boldsymbol{P}_t\right) = \max_{\{\boldsymbol{c}_t^k\}} \left\{ u_t^k\left(\boldsymbol{c}_t^k\right) \right\} \quad \text{s.t.} \quad \boldsymbol{P}'\boldsymbol{c}^k = e^k. \tag{29}$$

At the optimum, first-order conditions imply that

$$\boldsymbol{u}_c^k = U_e^k \boldsymbol{P} \quad \text{and} \quad -\boldsymbol{U}_p^k = U_e^k \boldsymbol{c}^k. \tag{30}$$

[16] Money is not modeled in this economy. However, if money were modeled in such a way that it entered the utility function like any other good, the analysis would not change. Stochastic properties of changes in the supply of money no doubt affect the covariances of goods' prices with aggregate real consumption; given this, monetary policy will affect the risk premium that the nominally riskless asset requires (see section 8). For such a model in a discrete-time economy, see Grauer and Litzenberger (1980).

From (30), at the optimum the percentage compensating variation in expenditure for a 1% change in a good's price is the good's budget share, i.e.,

$$\frac{-U^k_{pj}P_j}{U^k_e e^k} = \left.\frac{\partial \ln e^k}{\partial \ln p_j}\right|_{U^k} = \frac{P_j c^k_j}{e^k} = \alpha^k_j, \quad \forall j, \tag{31}$$

where α^k_j is k's budget share for good j. These are standard results.

An additional result that is later useful is the following, which arises from differentiating (30) with respect to expenditure:

$$\boldsymbol{U}^k_{pe} = -U^k_e\left[\partial \boldsymbol{c}^k/\partial e^k\right] - U^k_{ee}\boldsymbol{c}^k, \tag{32}$$

or

$$\mathbf{I}_P \boldsymbol{U}^k_{pe} = U^k_{ee}[T^k \boldsymbol{m}^k - e^k \boldsymbol{\alpha}^k], \tag{33}$$

where $\mathbf{I}_P$ is the diagonal matrix of goods prices, T^k_e is k's absolute risk tolerance for expenditure, $\boldsymbol{\alpha}^k$ is k's vector of budget shares, and $\boldsymbol{m}^k$ is k's vector of 'marginal budget shares'. That is, m^k_j is the fraction of an additional dollar of expenditure that would be spent by k on good j. The marginal and average budget share vectors are related by expenditure elasticities of demand, η^k_j, as follows: $m^k_j = \alpha^k_j \eta^k_j$. With this structure, the local validity of the Divisia price index is straightforward, as is shown in appendix 2.

In a multi-good economy, the price of a nominally riskless, T-period discount bond is equal in equilibrium to the expected marginal rate of substitution of nominal expenditure at time T for current nominal expenditure,

$$B_{t,T} = \mathrm{e}^{-r(t,T)(T-t)} = \frac{\mathrm{E}\left\{\mathrm{e}^{-\rho(T-t)} U^k_e\left(e^k_T, \boldsymbol{P}_T\right)\right\}}{U^k_e\left(e^k_t, \boldsymbol{P}_t\right)}, \tag{34}$$

which is identical in structure to the single-good relation. The Cox, Ingersoll and Ross (1985) proof that the instantaneous-maturity nominally riskless rate equals the negative of the expected rate of change of the marginal utility of wealth did not depend upon their assumption that there was only a single good. That fact, combined with the multi-good optimality condition that the marginal utility of nominal expenditure equals the marginal utility of wealth, permits us to express the riskless rate as the negative of the expected rate of change of $U^k_e(e^k, \boldsymbol{P}, t)$. Using Ito's Lemma to determine that expected rate of change, the instantaneously riskless nominal interest rate may be written as

$$r = \rho^k - \left(U^k_{ee}/U^k_e\right)\mu_{ek} - U^k_{eP}\mathbf{I}_P\boldsymbol{\mu}_P - \left(U^k_{eee}/2U^k_e\right)\sigma^2_{ek}$$

$$-\left(U^k_{eeP'}/U^k_e\right)\mathbf{I}_P V_{Pe} - \left(1/2U^k_e\right)\left[\mathbf{U}^k_{ePP}\Box\mathbf{I}_P\mathbf{V}_{PP}\mathbf{I}_P\right], \quad \forall k. \tag{35}$$

In (35), μ_{ek} and σ^2_{ek} are the drift and variance rates for k's nominal expenditure rate, V_{Pe} is the vector of covariances of k's expenditure with consumption-goods' percentage price changes, and $\mathbf{V}_{PP}$ is the variance–covariance matrix of goods' percentage price changes.

Aggregation of the drift components of individuals' expenditure rates and budget share vectors in (35) is straightforward, but aggregation of the variance terms is not. To see this, multiply (35) by T^k_e, sum across individuals, divide by $T^m_e = \sum_k T^k_e$, and substitute (33) for U^k_{eP} to get

$$r = \sum_k \left(T^k_e/T^m_e\right)\rho^k + \left(1/T^m_e\right)\mu_E + \left(1/T^m_e\right)\sum_k \left[T^k_e \boldsymbol{m}^k - e^k\boldsymbol{\alpha}^k\right]'\boldsymbol{\mu}_P$$

$$+\left[\left(1/2T^m_e\right)\left[\sum_k \left(u^k_{eee}/U^k_{ee}\right)\sigma^2_{ek}\right] + \left(1/T^m_e\right)\left[\sum_k \left(U^k_{eeP'}U^k_{ee}\right)\mathbf{I}_P V_{Pe}\right]\right.$$

$$\left. + \left(1/2T^m_e\right)\left[\sum_k \left(\mathbf{U}_{ePP}/U^k_{ee}\right)\Box\left(\mathbf{I}_P\mathbf{V}_{PP}\mathbf{I}_P\right]\right]\right]. \tag{36}$$

The first term of (36) is just a risk tolerance weighted average of individuals' pure rates of time preference, which will be denoted ρ^m. The second term reflects the expected growth of aggregate nominal expenditure, the third term picks up two offsetting inflation effects, and the final bracketed expression, which will be denoted $F(\mathbf{V})$, is a function of preferences and the variances and covariances of expenditures and prices.

The expected inflation effects aggregate cleanly. First, since $\boldsymbol{\alpha}^k$ is k's vector of budget shares, $\sum_k \boldsymbol{\alpha}^k e^k = \boldsymbol{\alpha}^m E$, where E is aggregate nominal expenditure and $\boldsymbol{\alpha}^m$ are the fractions of aggregate expenditure spent on the various goods in the economy. Next, it has been shown [Breeden (1979)] that one dollar additional aggregate expenditure is optimally allocated (holding prices constant) to individuals in proportion to their risk tolerances, i.e., T^k_e/T^m_e goes to individual k. Since individual k spends an additional dollar on goods in the marginal proportions $\boldsymbol{m}^k$, $\sum_k (T^k_e/T^m_e)\boldsymbol{m}^k$ gives the aggregate marginal budget share vector, $\boldsymbol{m}^m$. That is, $\boldsymbol{m}^m$ represents the incremental aggregate expenditures on the various goods that occur when one dollar additional aggregate nominal expenditure is optimally allocated across individuals and then optimally spent on goods by the individuals. Goods that have aggregate expenditure elasticities in excess of one will have marginal expenditure shares that exceed their average expenditure shares, whereas those with low expenditure elasticities will have $m_j^m < \alpha_j^m$. Finally, the products $\boldsymbol{m}^{m\prime}\boldsymbol{\mu}_P$ and $\boldsymbol{\alpha}^{m\prime}\boldsymbol{\mu}_P$ represent the expected inflation rate with aggregate marginal budget shares and aggregate average budget shares, respectively.

Given these arguments and definitions, the instantaneous nominally riskless interest rate may be written as

$$r = \rho^m + (E/T_e^m)\left[(\mu_E/E) - \boldsymbol{\alpha}^{m\prime}\boldsymbol{\mu}_P\right] + \boldsymbol{m}^{m\prime}\boldsymbol{\mu}_P + F(\mathbf{V}). \tag{37}$$

This expression for the riskless rate is intuitive in light of the single-good analyses. The aggregate pure rate of time preference is the first term. The expected percentage growth rate of aggregate real consumption is positively related to the riskless rate, multiplied by an aggregate measure of relative risk aversion. Appendix 2 demonstrated that the percentage change in aggregate nominal expenditure, less an inflation rate computed with aggregate budget shares as weights, is an apt aggregate quantity index, so the second term in (37) is analogous to the single-good economy's growth term.

The 'Fisher effect' term in the nominally risk-free rate is interesting, in that it is an inflation rate measured with aggregate marginal expenditure shares. Thus, while the Personal Consumption Expenditure Deflator is appropriate for measuring real consumption growth (since it uses aggregate value weights), it is theoretically inappropriate as an 'add-on inflation premium'. If homotheticity assumptions are made, the marginal and average aggregate inflation rates are identical, but that is an unlikely case.

A simple, heuristic explanation for the marginally weighted inflation rate is the following. Interest rates reflect optimal consumption–savings decisions by equating bond prices to all individuals' optimal expected marginal rates of substitution of future dollars for current dollars. At the margin, the evaluation of dollars now versus future dollars is an evaluation of the bundle of goods sacrificed at the margin today for an increment to the future bundle. The effects of price level changes on optimal marginal rates of substitution should be weighted in proportion to the quantities sacrificed and the future quantities gained, which are the marginal consumption bundles. Note that since the time period is infinitesimal in the period covered by (37), and since the Ito assumption makes the marginal bundles continuous, the relevant current and future marginal bundles are the same for (37). For discrete horizons, a similar marginal analysis could be done, but the marginal bundle today would not be the same as the marginal bundle at future dates.

In the discussion thus far, the variance and covariance terms have all been lumped into $F(\mathbf{V})$. Before these terms are explored, note that under certainty these terms are zeroes and this analysis is complete. Also note that in a model with logarithmic utility and homothetic indifference curves, the $(E/T_e^m)\boldsymbol{\alpha}^m$ and the $\boldsymbol{m}^m$ terms cancel, leaving only the growth of aggregate nominal expenditure from the middle term of (37). That case corresponds to the uncertain inflation model examined by Cox, Ingersoll and Ross (1985, sect. V).

8. Interest rates and consumption uncertainty with inflation

With no restrictions on individuals' preferences for goods, the analysis of the relation between the nominally riskless interest rate and goods' price level *uncertainties* [eq. (37)] is extremely complex. Although some qualitative features of that general relation will be noted later, a simple and clean (negative) relation of the riskless interest rate and the variance of aggregate real consumption is not easily derived. These difficulties arise due to the changing average and marginal consumption bundles in the general case, which affect the coefficients of the variance and covariance terms in the riskless rate relation. Another way to see the problem is through eq. (33), which shows the marginal utility may be either upward- or downward-sloping in commodities' prices. Thus, marginal utility is not monotonically related to real consumption for general utility [there are offsetting price and quantity effects in (33)]. The mean preserving spread approach that worked with a single good in the comparative statics on variance gives ambiguous results in the general case.

To circumvent these problems and proceed with the analysis of the effects of uncertainty, this section assumes that individuals have Cobb–Douglas (power) utility functions for goods. These utility functions are well-known to have unitary income elasticities of demand for all goods, unitary own-price elasticities of demand, and zero cross-price elasticities of demand. Optimal budget shares are non-stochastic, which permits computation of an invariant price index. Many authors have used this assumption in multi-good analyses, despite its lack of generality.[17] The specific assumption is that k's instantaneous utility for the goods bundle $\boldsymbol{c}^k$ is

$$u^k(\boldsymbol{c}^k, t) = \mathrm{e}^{-\rho t}\left[\prod_{i=1}^{N} c_i^{\alpha(i,k)}\right]^{1-\gamma}, \tag{38}$$

where $\sum_i \alpha(i,k) = 1$. For an expenditure rate of e^k and a price vector $\boldsymbol{P}$ the individual's optimal consumption rates [which maximize (38)] are

$$c_i^k = \alpha_i^k e^k / P_i \quad \text{or} \quad \alpha_i^k = P_i c_i^k / e^k, \quad \forall i. \tag{39}$$

These demand functions are well-known and have the properties noted earlier.

[17] Grauer and Litzenberger (1979), Long and Plosser (1983), and Cox, Ingersoll and Ross (1985) all used the Cobb–Douglas formulation, with LP and CIR also using a logarithmic utility assumption.

Given these optimal demand functions, k's indirect utility function for nominal expenditure and price vector $\boldsymbol{P}$ is

$$U^k(e^k, P, t) = \mathrm{e}^{-\rho t}\left[\prod_{i=1}^{N}\left(\alpha_i^k e^k/P_i\right)^{\alpha(i,k)}\right]^{1-\gamma} = A\,\mathrm{e}^{-\rho t}\left[e^k/I^k\right]^{1-\gamma}, \tag{40}$$

where

$$I^k = \prod_{i=1}^{N} P_i^{\alpha(i)/\Sigma\alpha(j)} \quad \text{and} \quad A = \left[\prod_{i=1}^{N}\alpha^{\alpha(i,k)}\right]^{1-\gamma} \tag{40$'$}$$

The individual's consumer price index, I^k, does not depend upon the level of expenditure, which is mathematically useful, but not economically plausible. It has constant elasticities with respect to commodities' prices, which are a result of the optimality of constant budget shares across states of the world. Utility is monotonically increasing and strictly concave in real expenditure, $e^{*k} = e^k/I^k$, and relative risk aversion for fluctuations in real expenditure is the constant γ.

At the optimum, the nominally riskless interest rate is minus the expected rate of growth of the marginal utility of a dollar. For analytical convenience, let that marginal utility function be denoted by $u'(\ln\tilde{e}, \ln\tilde{I}, t)$, where the individual superscripts are suppressed. From (40), letting subscripts of u' denote partial derivatives, we have

$$u' = A(1-\gamma)\,\mathrm{e}^{-\rho t}\mathrm{e}^{-\gamma\ln e}\mathrm{e}^{(\gamma-1)\ln I}. \tag{41}$$

The nominal rate is $r = -\mu_{u'}/u'$, which Ito's Lemma for u' gives as[18]

$$\begin{aligned} r = {} & \rho + \gamma\mu_{\ln e} - (\gamma-1)\mu_{\ln I} \\ & - \tfrac{1}{2}\left[\gamma^2\sigma^2_{\ln e} + 2\gamma(1-\gamma)\sigma_{\ln e,\ln I} + (\gamma-1)^2\sigma^2_{\ln I}\right]. \end{aligned} \tag{42}$$

This is just eq. (40) for the special case of Cobb–Douglas utility functions. This equation can be re-arranged to a much more intuitive form:

$$r - \mu_I/I + \sigma^2_{\ln I} = \rho + \gamma\mu_{\ln e^*} - (\gamma^2/2)\sigma^2_{\ln e^*} + \gamma\sigma_{-\ln I,\ln e^*}, \tag{43}$$

[18] Partial derivatives in log form are simple:

$$u'_{\ln e} = -\gamma u', \quad u'_{\ln e,\ln e} = \gamma^2 u', \quad u'_{\ln I} = (\gamma-1)u',$$
$$u'_{\ln I,\ln I} = (\gamma-1)^2 u', \quad u'_{\ln e,\ln I} = \gamma(1-\gamma)u'.$$

where

$$\mu_I/I = \mu_{\ln I} + \sigma^2_{\ln I}/2,$$

$$\mu_{\ln e^*} = \mu_{\ln e} - \mu_{\ln I},$$

$$\sigma^2_{\ln e^*} = \mathrm{var}(\ln(e/I)) = \sigma^2_{\ln e} - 2\sigma_{\ln e,\, \ln I} + \sigma^2_{\ln I}.$$

Fischer (1974) showed that the LHS of (43) is the expected real return on the nominally riskless bond. Thus, this says that, in equilibrium, the expected real return on the nominally riskless assets equals (1) the rate of pure time preference, plus (2) the expected growth rate of real expenditure, multiplied by relative risk aversion, minus (3) the variance of real expenditure, multiplied by RRA squared, and plus (4) the risk premium for the nominally riskless bond, which is proportional to the covariance of the real return on the nominal bond with real expenditure (as in the consumption-oriented CAPM).

Consider this expression for the nominal rate in relation to that of eq. (37), which was derived with very general consumption preferences. The drift terms in the general case were easy to explain, so the Cobb–Douglas assumption is unnecessary for understanding those terms. The variance term in (37) is seen to include at least two effects – the negative relation of the riskless rate to the uncertainty of real expenditure and the positive or negative risk adjustment for the real consumption beta of the nominally riskless asset. In general, the variance term in (37) also has terms that reflect uncertain changes in budget shares (and inflation measures) as prices and expenditure fluctuate, as well as changes in relative risk aversion. These terms are all zeroes with the Cobb–Douglas preferences of this section.

In this economy, a real riskless asset maturing at time t for individual k is one that pays k's price index at that time, $\tilde{I}^k_t = \prod \tilde{P}_i^{\alpha(i,k)}(t)$. Such an asset has a real payoff of one dollar at time t in all events. From Ito's Lemma, the stochastic component of the nominal return on this real riskless asset when it is at maturity is the following linear combination of commodity prices' stochastic components:

$$\sum_i \left(\partial I^k/\partial P_i\right)\sigma_{P_i}\mathrm{d}z_{P_i} = \sum_i I_k \alpha^k_i \sigma_{P_i}\mathrm{d}z_{P_i}/P_i = \sum_i I^k \alpha^k_i \sigma_{\ln P_i}\mathrm{d}z_{P_i}. \tag{44}$$

Given this, the equilibrium expected nominal return on the real riskless asset can be found from the return on the nominally riskless asset (43) and from the first-order conditions for an optimal portfolio [eq. (12′)]. Letting w^{*k} be individual k's augmented vector of investments in both productive processes and financial assets, the optimal portfolio is

$$w^{*k}Q^k = \left[-J^k_Q/J^k_{QQ}\right]\mathbf{V}^{-1}(\boldsymbol{\mu} - \boldsymbol{r}) - \mathbf{V}^{-1}\mathbf{V}_{qa,s}\left[J^k_{sQ}/J^k_{QQ}\right], \quad \forall k. \tag{12′}$$

Multiplying this by $J^k_{QQ}\mathbf{V}$ and re-arranging terms gives

$$\begin{aligned}\boldsymbol{\mu}-\boldsymbol{r}&=\left(-1/J^k_Q\right)\left[\mathbf{V}_{qa,Q}J^k_{QQ}+\mathbf{V}_{qa,s}J^k_{sQ}\right]\\&=\operatorname{cov}\left(\tilde{\boldsymbol{r}}_a,\ -\mathrm{d}\tilde{J}^k_Q/J^k_Q\right)=\operatorname{cov}\left(\tilde{\boldsymbol{r}}_a,\ -\mathrm{d}\tilde{U}^k_e/U^k_e\right),\quad \forall k,\end{aligned}\tag{45}$$

where $\tilde{\boldsymbol{r}}_a$ is the vector of realized nominal returns on risky investments. This is the familiar result that the equilibrium expected nominal excess returns on all assets are equal in equilibrium to the negatives of their covariances with the rate of change of marginal utility. Furthermore, since for an optimal policy the marginal utility of nominal expenditure equals the marginal utility of nominal wealth, (45) may be rewritten with J^k_Q replaced by U^k_e or U'^k.

Let μ_r be the expected instantaneous nominal return on the real riskless asset, and let r_r be its realized value. Given the Cobb–Douglas preference assumption, Ito's Lemma and (41) and (44) may be substituted into (45) to give μ_r:

$$\begin{aligned}\mu_r-r&=\operatorname{cov}\left[\tilde{r}_r,\quad -(1/u')\left\{u'_{\ln e}(\mathrm{d}\ln\tilde{e})+u'_{\ln I}(\mathrm{d}\ln\tilde{I})\right\}\right]\\&=\operatorname{cov}\left[\boldsymbol{\alpha}^k(\mathrm{d}\ln\tilde{\boldsymbol{P}}),\quad \gamma(\mathrm{d}\ln\tilde{e})+(1-\gamma)(\mathrm{d}\ln\tilde{I})\right]\\&=\sigma^2_{\ln I}+\gamma\sigma_{\ln I,\ln e^*}.\end{aligned}\tag{46}$$

Combining (46) with (44), the equilibrium expected nominal return on the real riskless asset is

$$\mu_r-\mu_I/I=\rho+\gamma\mu_{\ln e^*}-\left(\gamma^2/2\right)\sigma^2_{\ln e^*}.\tag{47}$$

As in the single-good case, the real riskless rate is positively related to pure time preference, and to the expected rate of growth of real expenditure, and is negatively related to the variance of real expenditure.

Note, however, that the real riskless rate is not observed, so the return on a nominally riskless bond is used in empirical tests. For that, (43) applies, which includes a risk premium for the real consumption risk of the nominally riskless asset. Since inflation is typically believed to be related to the growth rate of real consumption, the risk premium of the nominally riskless asset may be non-trivial. The relation of inflation to the real growth of the economy may be nonstationary, as can be easily illustrated. If a Phillips curve relates inflation and unemployment (pre-1973?), then inflation is likely to be high when real consumption is high, resulting in a negative real consumption beta for the nominally riskless assets (Treasury bills). This negative beta implies a lower equilibrium real return on Treasury bills than on purchasing power bonds. In contrast, recent experience [see Fama (1982)] has been that inflation is negatively related to real movements in the economy. If that were expected, then the real consumption betas for nominally riskless assets are positive, which results in equilibrium real returns on them that are in excess of those on purchasing power bonds.

9. Conclusion

This paper derived fairly general relations of consumption, production and interest rates in both a discrete-time state preference economy and in a continuous-time economy, using time-additive preferences throughout. Riskless interest rates were shown to be positively related to the expected growth rate of aggregate consumption, with a coefficient that is a measure of aggregate relative risk aversion. Rates were negatively related to the variance rate of aggregate consumption. In a separate equation, riskless rates were positively related to the expected productivity for optimal aggregate investments, with a coefficient of unity. They were negatively related to the variance of optimal aggregate production. The riskless interest rate was shown to be related to the autocorrelation (if any) in production, with the coefficient being related to the degree of relative risk aversion. Some of these results have been derived in less general models by other authors [notably Rubinstein (1976, 1981) and Cox, Ingersoll and Ross (1985)]. The focus in this paper was upon generalizing and explaining these results in simple economic terms and showing how optimal behavior leads to two separate interest rate relations to give the same term structure (since they must hold simultaneously in equilibrium).

Uncertainty in the economy arose from uncertain production and random technological change. The modeling included no stationarity assumptions for most results, so the uncertainty representation was quite general. Thus, consistent dynamic analysis of the term structure under uncertainty is well justified for these equations. A non-exhaustive discussion was presented of possible rational movements of interest rates during a business cycle.

There are two important extensions that are left for subsequent research. First, preferences exhibiting time complementarity in the utility of consumption were not permitted. Modeling of non-additive preferences will significantly affect marginal rates of substitution of consumption at one date for consumption at another date. Therefore, relaxation of the time-additive assumption could have a significant effect on the sizes of some of the effects (although the general relations derived here should remain intact). See Hansen and Singleton (1983), Dunn and Singleton (1983), and Ronn (1983) for results with particular forms of non-additive preferences.

A second important area for future research is on dropping the assumption (in the continuous-time model) that production plans are fully adjustable at each instant. In the continuous-time economy, there is no formal modeling of the 'time to build', as in Kydland and Prescott's (1982) work. Since the discrete approximations to the term structure (section 2) can handle time-to-build analyses, and since the results of that section are similar to those in the remainder of the paper, the basic relations should remain intact with more formal modeling of time to build. A virtue of this paper, in contrast to the early works in the non-separable utility and time-to-build areas (which are just developing), is that the general aggregation problem has been attacked with some success. Simple and intuitive relations of aggregate production, aggregate consumption and interest rates were derived.

Appendix 1

Outline of the Proof of Theorem 1: With appropriate convexity assumptions [see Debreu (1959, ch. 6)], there is a correspondence between the production plans and consumption allocations that a central planner would choose and those of a competitive equilibrium with complete capital markets. Characteristics of the central planner's optimal choices also apply to a competitive equilibrium with complete (or Pareto-optimal) capital markets. With assumptions (A.1) and (A.2), each individual's optimal consumption at each date is a function of only aggregate consumption and time. The probability distribution for aggregate consumption completely describes the probability distributions for all individuals' consumptions and, as a result, fully describes each individual's expected utility and marginal utility levels. Thus, if the (conditional) probability distribution for optimal aggregate consumption at all future dates is the same at time t in states θ_1 and θ_2, then each individual's expected utility of lifetime consumption is the same in those two states, and the probability distribution of marginal rates of substitution across dates is the same as seen in the two states. These results are useful in the proof.

For a discrete-time economy with a final date of T, consider the problem moving backwards from T. At any date t, let $S(t)=\{\theta_{1t},\theta_{2t},\dots,\theta_{n(t)t}\}$ be the set of fully descriptive (Arrow–Debreu) states that are possible, where the θ_{jt} are all scalars. The joint probability distribution of all future returns from investments in production processes in any state θ_{1t} is fully described by the vector $s_t(\theta_t)$. Since at T the entire amount of the good will be consumed, all states at $T-1$ that have the same total amount of the good, $Q(\theta, T-1)$, and the same production opportunity set, $s(\theta, T-1)$, have the same objective function and feasible set and, hence, the same optimal production and consumption plans. In particular, they have the same probability distribution at $T-1$ for everyone's consumption at T. From this argument, everyone's expected utility of (remaining) lifetime consumption as seen at $T-1$ depends only upon aggregate supplies at that time and the production opportunity set, s, i.e., $J^k(T-1,\theta)=J^k(Q(\theta), s(\theta), T-1)$.

Next, consider the situation at $T-2$. If two states at $T-2$ have the same production opportunity set, then the joint distribution for $\{Q(T-1)|$

$q(T-2), s(T-2)\}$ must be the same in both states [where $q(T-2)$ is the vector of inputs at $T-2$ to the various production processes]. Note that the production opportunity set describes not only current production possibilities, but also future production possibilities and the joint distribution of current and future possibilities. As at $T-1$, if the planner has the same aggregate supplies and production possibilities in two states, then the planner faces the same objective and is subject to the same constraints on probability distributions for lifetime consumption that can be given to individuals. Thus, assuming uniqueness of the solution to this problem, individuals' expected utilities at $T-2$ for lifetime consumption are functions of only $Q(T-2)$ and $s(T-2)$. This argument can be iterated back to the initial date, deriving k's expected utility as a function of only aggregate supplies, the production opportunity set, and time.

Consider the roles played by the two assumptions. With heterogeneous beliefs, allocations depend upon the distribution of beliefs, as well as on technology and supplies. If individuals have utility functions that are not time-additive, then past and future consumption will affect the expected utility and expected marginal utility of current consumption. Thus, even with the same aggregate wealth and production opportunities in two states at the same date, the optimal consumption and production plans may vary.

In our model, current optimal consumption and the probability distribution for future consumption depends only upon $\{Q(t), s(t), t\}$. The last step is to show that the optimal wealth allocation in the competitive equilibrium is also completely determined by $\{Q, s, t\}$. Let $S(Q, s, t)$ be the set of states at t that have aggregate supply Q and opportunity set s. From the budget constraint, k's wealth at time τ in state ξ is

$$W^k_{\tau\xi}=\sum_{t>\tau}\sum_{\theta\in S(t)} c^k(Q_{t\theta}, s_{t\theta}, t)\phi_{t\theta|\tau\xi}$$

$$=\sum_{t>\tau}\sum_{\theta\in S(t)} c^k(Q_{t\theta}, s_{t\theta}, t)\left[\pi_{t\theta|\tau\xi}u'^k(Q_{t\theta}, s_{t\theta}, t)/u'^k(Q_{\tau\xi}, s_{\tau\xi}, \tau)\right]$$

$$=\sum_{t>\tau}\sum_{S\{Q,s,t\}} c^k(Q, s, t)\left[\pi_{\{Q,s,t\}|\tau\xi}u'^k(Q, s, t)/u'^k(Q_{\xi}, s_{\xi}, \tau)\right]$$

$$\times\sum_{\theta\in S(Q,s,t)}\pi_{t\theta|Q,s,t}$$

$$=\sum_{t>\tau}\sum_{S\{Q,s,t\}} c^k(Q, s, t)\left[\pi_{\{Q,s,t\}|\tau\xi}u'^k(Q, s, t)/u'^k(Q_{\tau\xi}, s_{\tau\xi}, \tau)\right].$$

Since the last expression is the same for all states at τ that have the same aggregate supply and opportunity set, $W^k_{\tau\xi}=f\{Q, s, t\}$. Q.E.D.

Appendix 2

One well-known price index problem is that the mix of goods consumed varies with the level of expenditure, so the measurement of inflation also varies with expenditure. A second problem is aggregation of individuals' real consumptions, or alternatively, with the construction of a meaningful index of inflation for the economy. Even if all expenditure elasticities are assumed to be unity, different individuals will have different vectors of budget shares and, hence, different price indices. Is there any meaning to aggregate real consumption and a price index with aggregate budget shares?

With both nominal expenditure and consumption-goods prices following Ito processes, Ito's Lemma implies that the current utility of consumption, $u^k(e^k, \boldsymbol{P})$, has the stochastic differential

$$\mathrm{d}u^k = \left[u_t^k + u_e^k \mu_{ek} + \boldsymbol{u}_P^k \boldsymbol{\mu}_P + \tfrac{1}{2} \begin{bmatrix} u_{ee}^k & \boldsymbol{u}_{eP}^k \\ \boldsymbol{u}_{Pe}^k & \mathbf{u}_{PP}^k \end{bmatrix} \square \begin{bmatrix} \sigma_{ek}^2 & \boldsymbol{V}_{eP}^k \\ \boldsymbol{V}_{Pe}^k & \mathbf{V}_{PP} \end{bmatrix} \right] \mathrm{d}t$$

$$+ u_e^k \sigma_{ek} \, \mathrm{d}z_{ek} + \boldsymbol{u}_P^k \boldsymbol{\sigma}_P \, \mathrm{d}\boldsymbol{z}_P. \tag{A.1}$$

Letting the unexpected local changes in nominal expenditure and prices be denoted by $\widetilde{\mathrm{d}e}^k$ and $\widetilde{\mathrm{d}\boldsymbol{P}}$, the unexpected change in instantaneous utility is

$$\widetilde{\mathrm{d}u}^k = u_e^k \left[\widetilde{\mathrm{d}e}^k + \left(\boldsymbol{u}_P^{k\prime} / u_e^k \right) \left(\widetilde{\mathrm{d}\boldsymbol{P}} \right) \right]$$

$$= u_e^k \left[\left(\widetilde{\mathrm{d}e}^k / e^k \right) - \boldsymbol{\alpha}^{k\prime} \left(\mathbf{I}_P^{-1} \widetilde{\mathrm{d}\boldsymbol{P}} \right) \right] e^k. \tag{A.2}$$

Divisia's computation of real consumption's percentage change is the percentage change in k's nominal expenditure, less a budget share weighted average of the percentage changes in goods prices. This is precisely the bracketed term in (A.2). Thus, k's utility of current consumption in alternate $\{e^k, \boldsymbol{P}\}$ states is one-to-one with the Divisia measure of percentage change in k's real consumption. Individual k has a higher utility of current consumption if and only if k's real consumption grows, given inflation measured by a value-weighted price index (similar to the PCE deflator). Note that this did not require unitary expenditure elasticities.

A more interesting result is that computed changes in *aggregate* real consumption have economic content, even with diversity and non-homotheticity of consumption preferences across individuals. To see this, define as a reference state that state where each individual's expenditure is as expected and where the commodity price vector is at its expected level. Define the computed percentage change in aggregate real consumption as the percentage change in aggregate nominal expenditure, $E=\sum_k e^k$, less an inflation rate computed with the aggregate expenditure shares of goods as weights, $\boldsymbol{\alpha}^m = \sum_k e^k \boldsymbol{\alpha}^k / E$:

$$\mathrm{d}E^* = \mathrm{d}E - E\boldsymbol{\alpha}^{m\prime}\left[\mathbf{I}_P^{-1}\,\mathrm{d}\boldsymbol{P}\right]. \tag{A.3}$$

Let $\overline{\mathrm{d}E}$ and $\overline{\mathrm{d}\boldsymbol{P}}$ be the expected changes in aggregate expenditure and goods prices. Consider now a state where aggregate real expenditure grows at a rate g percent more than its expected growth, i.e.,

$$\mathrm{d}E_s - E\boldsymbol{\alpha}^{m\prime}\left[\mathbf{I}_P^{-1}\,\mathrm{d}\boldsymbol{P}_s\right] = \overline{\mathrm{d}E} - E\boldsymbol{\alpha}^{m\prime}\left[\mathbf{I}_P^{-1}\,\overline{\mathrm{d}\boldsymbol{P}}\right] + gE\,\mathrm{d}t. \tag{A.4}$$

In such a state, goods can be allocated so that each individual has higher real consumption (as she computes it) than in the reference state. One such allocation gives each individual the following:

$$\mathrm{d}e_s^k - e^k\boldsymbol{\alpha}^k\left[\mathbf{I}_P^{-1}\,\mathrm{d}\boldsymbol{P}_s\right] = \overline{\mathrm{d}e}^k - e^k\boldsymbol{\alpha}^k\left[\mathbf{I}_P^{-1}\,\overline{\mathrm{d}\boldsymbol{P}}\right] + ge^k\,\mathrm{d}t. \tag{A.5}$$

Aggregating (A.5) across individuals gives (A.4), which shows that the allocation is feasible. Thus, if aggregate real consumption grows at a greater rate than expected, then there exists a Pareto-superior allocation of consumption goods, relative to the expected allocation.

It is also necessary that aggregate real consumption be above the expected for there to exist a Pareto-superior allocation relative to the expected allocation. To see this, assume the contrary, i.e., that there is a Pareto-superior allocation in a state s that has lower than expected growth of aggregate real consumption. For each individual to view his allocation as superior to the expected, (A.2) implies

$$\mathrm{d}e_s^k - e^k\boldsymbol{\alpha}^{k\prime}\left[\mathbf{I}_P^{-1}\,\mathrm{d}\boldsymbol{P}_s\right] > \overline{\mathrm{d}e}^k - e^k\boldsymbol{\alpha}^{k\prime}\left[\mathbf{I}_P^{-1}\,\overline{\mathrm{d}\boldsymbol{P}}\right], \quad \forall k. \tag{A.6}$$

However, aggregation of this across individuals implies that the growth of aggregate real consumption exceeds its expected growth, if all individuals prefer the state s allocation. This contradicts the hypothesis and demonstrates the necessity result.

Restated, the result is that local aggregate real consumption changes are valid local measures of changes in an economic quantity index for the economy. The larger the change in aggregate real consumption, the larger the feasible change in each individual's quantity index. This is true locally even with diverse and non-homothetic consumption preferences, which helps to explain why eq. (37) holds quite generally for the nominally riskless rate.

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