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Futures Markets and Commodity Options: Hedging and Optimality in Incomplete Markets

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This paper examines the allocational roles of futures markets and commodity options in multi-good and multi-period economies. In a continuous-time model with time-additive utilities and homogeneous beliefs, trading in "unconditional" futures contracts, the market portfolio and a riskless asset gives any Pareto-optimal allocation. Individuals' optimal holdings of futures contracts in the continuous-time model are related to their consumption bundles and to their risk tolerances. It is shown that both hedging and "reverse hedging" behavior are possible. In the general model with discrete trading, options on portfolios of commodity options are shown to permit any unconstrained Pareto-optimal allocation. *Journal of Economic Literature* Classification Numbers: 024, 026, 313.

I. INTRODUCTION

Recent works by Hakansson [21–23], Ross [32] and Breeden and Litzenberger [8] have considered alternative structures of capital markets that are considerably less extensive than a complete Arrow–Debreu securities market, but permit all unconstrained Pareto-optimal allocations of time-state contingent consumption claims, within the contexts of their models. A difficulty with their results is that they do not explicitly consider a multi-good economy, which has greater complexity of efficient allocations than does the single-good model.[1] This paper demonstrates that efficient allocations in a multi-good economy can be attained with a limited number of markets that include "unconditional" futures contracts or commodity options.[2]

The role of contingent commodity contracts in the allocation of goods under uncertainty has been amply demonstrated by Debreu [13]. However,

[1] As Ross's model assumed that the complete state space must be spanned, it includes a multi-good economy as a special case; however, the number of options necessary for spanning in that model is much greater than in the other, more recent papers referenced.

[2] Throughout this paper the focus is upon *ex ante* Pareto optimality, rather than upon the *ex post* optimality of Starr [35].

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contingent-claims markets are much less prevalent than would appear to be optimal. To quote a recent paper by Townsend [36, p. 54]:

> In particular, the existence of futures or forward markets in which unconditional rather than contingent claims are traded is regarded by some as a phenomenon in need of an explanation, and by others as prima facie evidence of some inefficiency.

Townsend then proves that if there are as many linearly independent spot commodity prices as there are states of the world, then unconditional forward contracts have the spanning property that Arrow–Debreu securities have, and they therefore constitute a Pareto-optimal capital market. Unfortunately, this is a very weak theorem in answer to the efficiency question raised in the quotation, since there surely are more economic states of the world than there are commodities. For example, with just two commodities that can each have prices of \$1 or \$2, four states of the world are necessary to provide a complete description of these prices: $\{(\$1, \$1), (\$1, \$2), (\$2, \$1), (\$2, \$2)\}$. Two unconditional futures contracts cannot span these state payoffs. Thus, the efficiency question remains unanswered for many interesting cases. This paper (Section III) shows that, with continuous trading in unconditional futures contracts, contingent futures contracts are not necessary for optimality of capital markets. Without continuous trading, options on specified portfolios of commodity options and aggregate nominal consumption are shown (Section VI) to comprise a Pareto-optimal capital market; again, contingent futures contracts are not necessary, as they are spanned by this capital market.

Sections IV and V, in a multi-good extension of Merton's [28] continuous-time economic model, derive individuals' long or short positions in the various futures contracts in terms of their measures of relative risk aversion, their consumption preferences and their reinvestment risks. These sections generalize and extend some of the hedging and 'reverse hedging" results that were presented by Merton [28], Long [26], Fischer [18], Dieffenbach [15], and Grauer and Litzenberger [19]. An individual's futures portfolio is considered as part of his overall asset portfolio. Since an individual chooses a portfolio that is mean–variance efficient in his real wealth, the futures portfolio is related to the effects of those holdings on the mean and variance of real wealth. It is shown that the variance-reducing effects of hedging (on real wealth) may be offset (in lifetime utility terms) by the mean-expanding impact of reverse hedging. Consistent with prior works, logarithmic utility is the dividing line between portfolios dominated by mean effects and those dominated by variance effects of futures. Throughout the hedging analysis, the effects of futures holdings on the mean and variance of lifetime utility are distinguished from their effects upon the mean and variance of current consumption.

II. Optimal Allocations in a Multi-good Multi-period Economy

In this section, characteristics of Pareto-optimal allocations of consumption goods in a multi-good multi-period economy are derived. In particular, using a preference assumption and an assumption on probability beliefs, it is shown that any individual k's optimal state-contingent consumption bundle of the N goods at time t, $\mathbf{c}_t^k = (c_{1t}^k, \ldots, c_{Nt}^k)'$, may be written as a function of (only) the vector of aggregate consumption goods at that time, $\mathbf{C}_t = \sum_k \mathbf{c}_t^k$. Furthermore, for any given date it is shown that all states of the world with the same level of aggregate nominal expenditure and consumption goods prices have the same optimal allocation of consumption goods to individuals. The results of these theorems are used in the next section to determine optimal capital market structures.

The restriction on individuals' preferences that is assumed is:

(A1) Each individual's von Neumann–Morgenstern utility function is time-additive and state-independent in terms of consumption bundles. Mathematically, individual k maximizes:

$$U^k(\mathbf{c}_1^k, \mathbf{c}_2^k, \ldots, \mathbf{c}_{T_k}^k) = \sum_t \sum_{s \in S_t} \pi_{ts}^k u^k(\mathbf{c}_{ts}^k, t), \tag{1}$$

where T_k is k's time of death, S_t is the set of possible states at time t, and π_{ts}^k is k's subjective probability for state s at time t. The utility function u^k is monotonically increasing and strictly quasi-concave.

In each of the paper's theorems, either homogeneous beliefs or one of the following two assumptions of conditionally homogeneous beliefs is used[3]:

(A2) Given the vector of aggregate consumption of all goods at date t, individuals agree upon the probabilities of states at date t.[4] That is, $\pi_{ts}^k = \pi_{t\mathbf{C}}^k \pi_{ts|\mathbf{C}}$, where $\pi_{t\mathbf{C}}^k$ is k's subjective probability for an aggregate consumption vector of $\mathbf{C}$ at time t, and $\pi_{ts|\mathbf{C}}$ is k's probability that state s occurs at time t, conditional upon an aggregate consumption vector of $\mathbf{C}$ at time t.

(A3) Given the $(N+1)$-vector of aggregate nominal consumption expenditure and goods prices at time t, $(E, \mathbf{P})$, individuals agree upon the probabilities of states at time t.

[3] Note that assumptions A2 and A3 are equivalent under the conditions of Theorems 1 and 2, if the allocation is optimal.

[4] The aggregate consumption vector is important information, but it is not a complete state description in the Arrow–Debreu sense. Given an aggregate consumption vector, states may, for example, differ in their descriptions of assets' payoffs, production possibilities, and probabilities of future consumption rates and asset returns.

The following two theorems characterize all Pareto-optimal allocations in this class of economies, which includes many of the models in the literature of financial economics.

THEOREM 1. *If assumptions* A1 *and* A2 *hold, then any unconstrained Pareto-optimal allocation of time-state contingent consumption goods is such that, at each date, all states with the same vector of aggregate goods consumption have the same allocation of goods to individuals. Furthermore, given* A1, A2 *is both necessary and sufficient for the theorem.*

THEOREM 2. *If assumptions* A1 *and* A3 *hold, then any unconstrained Pareto-optimal allocation of time-state contingent consumption goods is such that, at each date, all states with the same consumption-goods prices and aggregate nominal consumption expenditure have the same allocation of consumption goods and consumption expenditure to individuals. Furthermore, given* A1, A3 *is both necessary and sufficient for the theorem.*

The proofs of Theorem 1 and Theorem 2 are in Appendix A.

Theorem 1 implies that each individual's optimal vector of consumption goods at time t can be written as a function of (only) time and the aggregate vector of consumption goods at that time, i.e., $\tilde{\mathbf{c}}_t^k = \mathbf{c}^k(\tilde{\mathbf{C}}_t, t)$. A special case of this theorem is Breeden and Litzenberger's [8] single-good version, $\tilde{c}_t^k = c^k(\tilde{C}_t, t)$, which was a part of the consumption-oriented CAPM theory. Their single-good result implied that (with homogeneous beliefs) each individual's optimal consumption rate was locally perfectly correlated with every other person's and with the aggregate consumption rate. In this multi-good economy with homogeneous beliefs, marginal utilities of consumption dollars will be perfectly correlated. However, in a multi-good economy, the consumption rates of all individuals for a single good (such as cheese) will not be locally perfectly correlated, but will depend upon the aggregate supplies of *all* goods. The power of Theorem 1 is its statement that other economic variables such as interest rates, prospective returns on risky assets, and aggregate output affect individuals' optimal consumption rates if and only if they affect optimal aggregate consumption rates.

Theorem 2 is just the dual problem version of Theorem 1, stating the same basic result in terms of aggregate expenditure and the vector of goods' prices. The result is that individual k's optimal nominal expenditure rate at time t, $\tilde{e}_t^k$, can be written in terms of the aggregate nominal expenditure rate, the vector of consumption-goods prices and time, i.e., $\tilde{e}_t^k = e^k(\tilde{E}_t, \tilde{\mathbf{P}}_t, t)$.

From the result of Theorem 2 that the Pareto-optimal allocation of nominal consumption expenditure at date t depends only upon aggregate expenditure and consumption goods' prices at t, it is not surprising that futures contracts, forward contracts and commodity options have significant roles in risk allocation. However, with the generality and diversity of preferences permitted by the analysis, futures, forwards and options are *not* sufficient to span all possible Pareto-optimal allocations with simple buy-and-hold strategies. The risk sharing limitations of buying and holding these contracts are well-appreciated, so they will only be outlined prior to the next section's analysis of risk sharing with continuous trading.

Forward contracts may be viewed as having payoffs at maturity that are linear in (equal to) the underlying commodity's price at maturity. Consequently, these contracts and futures contracts for aggregate nominal consumption span the space of efficient allocations if and only if all individuals have optimal nominal consumption levels that are linear in aggregate consumption and consumption goods prices at each date, i.e., only if:

$$e_t^k(\tilde{E}, \tilde{\mathbf{P}}) = a_t^k + \sum_i b_{ti}^k \tilde{P}_{ti} + c_t^k \tilde{E}, \qquad \forall\, k, t. \tag{2}$$

Schrems [34], Ross [32], Breeden and Litzenberger [8], and Banz and Miller [3] showed that portfolios of call or put options on an asset with price $\tilde{P}_t$ can achieve any desired contingent payoff function, $f(\tilde{P}_t)$. Thus, portfolios of commodity options and options on aggregate consumption span the set of allocations that are additive (but not necessarily linear) functions of $(E, \mathbf{P})$, such as:

$$e_t^k(\tilde{E}_t, \tilde{\mathbf{P}}_t) = a_t^k + \sum_i f_{ti}^k(\tilde{P}_{ti}) + f_{te}^k(\tilde{E}_t). \tag{3}$$

Neither portfolios of futures contracts nor portfolios of commodity options can span the general space of efficient allocations as given by Theorem 2 since these simple portfolios do not capture the interactions among consumption goods prices and aggregate expenditure that determine the optimal allocation $\{e_t^k(\tilde{E}, \tilde{\mathbf{P}})\}$. For example, the payoff function $e^k(\tilde{P}_1, \tilde{P}_2) = \tilde{P}_1 \cdot \tilde{P}_2$ cannot be achieved by simple buying and holding of futures, forwards or commodity options on goods 1 and 2. The next sections examine more powerful sequential trading strategies.

III. Continuous Trading and the Optimality of Futures Markets

Now consider an economy with continuous trading and with all economic variables following diffusion processes, as in the models of Merton [28], Cox, Ingersoll and Ross [11], and Breeden [7]. In those models (as well as in discrete-time multi-period models), individual k uses dynamic programming to determine his indirect utility function for current wealth, W^k, and makes consumption-investment decisions for the current period based upon that function and the utility of current consumption. If the relevant characteristics of consumption, income and investment opportunities are stochastic over time and are represented by the "state" vector $\mathbf{s}$, then individual k's indirect utility function is written as $J^k(W^k, \mathbf{s}, t)$.

Individual k's demands for risky assets in the continuous-time multi-commodity model are[5]

$$\mathbf{w}^k W^k = T^k \underline{\mathbf{V}}_{aa}^{-1}(\boldsymbol{\mu} - \mathbf{r}) + \underline{\mathbf{V}}_{aa}^{-1} \underline{\mathbf{V}}_{as} \mathbf{H}_s^k, \tag{4}$$

where $\mathbf{w}^k$ is the $A \times 1$ vector of the individual's wealth shares invested in various risky assets, with $1 - \sum_i w_i^k = w_0^k$ as the fraction of wealth invested in the (nominally) riskless asset. The $A \times A$ incremental covariance matrix for instantaneous nominal rates of return on assets is $\underline{\mathbf{V}}_{aa}$, and $\underline{\mathbf{V}}_{as}$ is the $A \times S$ matrix of incremental covariances of assets' returns with the various state variables describing the investment, income and consumption opportunity sets. Individual k's absolute risk tolerance in terms of the indirect utility function is $T^k = -J_W^k / J_{WW}^k$ and k's relative risk tolerance is $T^{*k} = T^k / W^k$. The $S \times 1$ vector denoted by $\mathbf{H}_s^k = -J_{sW}^k / J_{WW}^k$ gives individual k's "hedging demands against adverse changes in the consumption-investment opportunity set," in Merton's [28] terminology. The instantaneous expected rates of return on risky assets are given by the $A \times 1$ vector, $\boldsymbol{\mu}$, and $\mathbf{r} = r \cdot \mathbf{1}$ where $\mathbf{1}$ is an $A \times 1$ vector of ones and r is the instantaneous riskless interest rate (in nominal terms).

Aggregating the demand vectors in (4) gives the market portfolio, $\mathbf{w}^M M = \sum_k \mathbf{w}^k W^k$. Individual k's risky asset holdings may be stated in terms of his amount in the market portfolio and his amounts in the S portflios having the highest correlation with the S state variables, respectively:

$$\mathbf{w}^k W^k = \frac{T^k M}{T^M} (\mathbf{w}^M) + \underline{\mathbf{V}}_{aa}^{-1} \underline{\mathbf{V}}_{as} \left(\mathbf{H}_s^k - \frac{T^k}{T^M} \mathbf{H}_s^M \right), \tag{5}$$

where $T^M = \sum_k T^k$ and $\mathbf{H}_s^M = \sum_k \mathbf{H}_s^k$.

Define a "futures contract of instantaneous maturity on state variable s_1" as an asset with a gross payoff in the next instant that is equal to the first state variable s_1. If s_1 is the price of corn, then this futures contract would have a gross payoff equal to the price of corn in the next instant, just as a standard futures contract *at maturity* can be regarded as having a gross payoff equal to the value of the underlying commodity. When an instantaneous-maturity futures contract is assumed to exist at all times for a variable like s_1, the implication is that there is at every instant a futures

[5] See Merton's [28] Eq. 16, which can be derived with many commodities in terms of a general state vector, X.

contract on s_1 that expires and another one created that matures in the instant.[6] This is analogous to the assumption that there always exists an instantaneous riskless discount bond, but that the interest rate on it changes stochastically over time.

Assume that there exist S such futures contracts of instantaneous maturity that have zero net supplies in aggregate, and whose returns (by definition) are perfectly correlated with changes in the various state variables.[7] The assumption that aggregate supplies of futures are zeroes (as is true with futures contracts traded on organized exchanges) implies that their fractions of the market portfolio are also zeroes. The fact that the payoffs of the futures contracts are the levels of the various state variables implies that $\underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as} = (\underline{\mathbf{I}}\,\underline{\mathbf{0}})'$, where $\underline{\mathbf{I}}$ is the $S \times S$ identity matrix and $\underline{\mathbf{0}}$ is the $S \times (A - S)$ matrix of zeroes.[8] The subvector of individual k's demands for futures contracts, $\mathbf{w}_s^k W^k$, is obtained by combining these two facts with (5)[9,10]:

$$\mathbf{w}_s^k W^k = \mathbf{H}_s^k - \frac{T^k}{T^M}\mathbf{H}_s^M. \tag{6}$$

From (5) and (6), it is seen that individual k's portfolio consists of $(T^k M/T^M)$ dollars in the market portfolio, S holdings of the form

[6] This constant dimension but changing securities market basis is discussed in Cox, Ingersoll, and Ross [11].

[7] An important point made by the referee is that if an opportunity set state variable (such as an expected inflation rate) is unobservable, then futures or options cannot be written for it. Theorem 3 demonstrates that if there is not a portfolio perfectly correlated with each state variable, then a reasonable economy can easily be found for which the allocation is not optimal. This is not such a significant problem in Theorem 2 since commodity prices and aggregate expenditure are more reasonably assumed to be observable.

[8] Breeden [7] demonstrated that $\underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as}$ has as its columns the portfolios of assets that have maximum correlations of returns with the various state variables, i.e., column j gives the portfolio that has the maximum correlation with state variable s_j. Given the "futures contracts" as defined, clearly these maximum correlation portfolios are holdings of only those futures contracts that correspond to the state variables (since they provide perfect correlations). Thus, $\underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as}$ must be diagonal. With suitable normalizations of state variables, the diagonal elements can all be set to unity.

[9] Standard futures contracts require no investment, which makes the vector of wealth fractions, $\mathbf{w}_s^k$, difficult to interpret. The interpretation of $\mathbf{w}_s^k$ is that this is the wealth share that should be invested in the portfolio of (1) the futures contract for s, and (2) a number of riskless bonds that pay in the next instant the current futures price multiplied by the "quantity" specified in the futures contract. In any case, the vector $\mathbf{w}_s^k$ divided by the vector of futures prices (one-by-one) gives the number of contracts that are optimally held (assuming each contract is for 1 unit).

[10] If there exists an asset with positive net supply that is perfectly correlated with a state variable, then (from (5)) k would hold $(T^k M/T^M)$ times its weight in the market portfolio, as well as its demand component in (6). For such assets, the subsequent analysis should be viewed as an analysis of their supplies and demands as deviations from the market portfolio holdings of them by individuals.

$(H^k_{s_j} - H^M_{s_j} T^k/T^M)$ in the futures contract for state variable j, and the remainder of wealth in the nominally riskless asset. Since the indirect utility function $J^k(W^k, \mathbf{s}, t)$ and its derivatives (which determine $\mathbf{H}^k_s$ and T^k) are all dependent upon the state of the world, each individual's portfolio weights will in general change over time in response to changing wealth, to changing consumption and investment opportunities and in response to life cycle considerations.

The paper's principal theorem on allocational efficiency is the following, Theorem 3. The theorem states that if individuals hold the market portfolio, the riskless asset, and the S futures contracts in the proportions just derived, then the resulting intertemporal allocation of consumption goods to individuals is an unconstrained Pareto-optimal allocation. The proof of the theorem demonstrates that the marginal rate of substitution of dollars at any one time and state for dollars at any other time and state is the same for all individuals, which is the criterion for an unconstrained Pareto optimum. The theorem is a global theorem, in the sense that it is true for allocations that are discrete distances apart in time, as well as for allocations that are at adjacent points in time; of course, continuous trading is assumed throughout the theorem. Furthermore, it is also shown that if there do not exist futures contracts or portfolios whose returns are locally perfectly correlated with state variables' changes, then the allocation will not be an unconstrained Pareto optimum for all preferences within the time-additive class, given that there is at least one stochastic state variable for opportunities.

THEOREM 3. *In the multi-good continuous-time economic model with individuals who have time-additive preferences as in* (A1) *and who have fully homogeneous beliefs, the following $S + 2$ funds (or any nonsingular transformation of them) are necessary and sufficient for the capital markets to permit all possible unconstrained Pareto-optimal allocations of time-state contingent consumption: an instantaneously riskless asset in nominal terms, the market portfolio, and S futures contracts (of instantaneous maturity) for the elements describing the consumption–income–investment opportunity set.*

Since the proof is not an obvious extension of proofs in the literature, it is presented in the text. Readers may skip to the next section, where the actual supplies and demands for futures contracts are examined, without losing the main points of the paper.

Proof. The criterion for an unconstrained Pareto optimum is that the marginal rate of substitution (mrs) of a unit of the numeraire ("dollars") between any two time-states be the same for all individuals. An optimal policy has the marginal utility of expenditure for individual k, $u^k_e(e^k, \mathbf{P}, t)$, equal to the marginal utility of wealth, $J^k_W(W^k, \mathbf{s}, t)$. Thus, the sufficiency part of the theorem may be shown by proving that (where θ is the state of the world at time t)

$$\frac{J_W^k(W^k(\theta), \mathbf{s}(\theta), t)}{J_W^k(W_{t_0}^k, \mathbf{s}_{t_0}, t_0)} = \frac{J_W^j(W^j(\theta), \mathbf{s}(\theta), t)}{J_W^j(W_{t_0}^j, \mathbf{s}_{t_0}, t_0)}, \qquad \forall \theta, \{j, k\}, t > t_0. \tag{7}$$

The equivalence of mrs's in (7) is shown in logarithm form by showing that

$$\ln J_W^k(\theta, t) - \ln J_W^k(t_0) = f(\theta, t), \qquad \forall k, \tag{8}$$

where $f(\theta, t)$ is independent of k. As seen today (at time t_0), the difference between the log of the marginal utility of wealth today and at a future time and state is given by Ito's stochastic integral as

$$\ln J_W^k(t) - \ln J_W^k(t_0) = \int_{t_0}^{t} \left[\frac{d \ln J_W^k(\tau)}{d\tau}\right] d\tau. \tag{9}$$

It will be shown that the Ito stochastic differential, $d \ln J_W^k$, is at all points in time the same for all k. This implies that the integral in (9) is the same for all k, which gives an $f(\theta, t)$ for which (8) is true, thereby proving sufficiency.

The drift and diffusion parameters for $d \ln J_W^k$ are given by Ito's lemma from those for dJ_W^k as follows:

$$dJ_W^k \equiv \mu_{J_W}^k \, dt + \sigma_{J_W}^k \, dZ^k$$

$$\Rightarrow d \ln J_W^k = \left[\frac{1}{J_W^k} \mu_{J_W}^k - \frac{1}{2(J_W^k)^2} (\sigma_{J_W}^k)^2\right] dt + \frac{1}{J_W^k} \sigma_{J_W}^k dZ^k. \tag{10}$$

Cox, Ingersoll, and Ross [11] have shown that $\mu_{J_W}^k / J_W^k = -r$, which is stochastic but the same for all k.[11] Thus, it remains to show only that the remaining two terms of (10) are the same for all individuals.

The third term in (10) is the locally stochastic component of the change in marginal utility. Since $J_W^k = J_W^k(W^k, \mathbf{s}, t)$, the stochastic movement in k's marginal utility can be derived from the stochastic movements in k's wealth and in the state vector for opportunities. Ito's lemma gives

$$\sigma_{J_W}^k dZ^k = J_{WW}^k [\sigma_W^k dZ_W^k] + \mathbf{J}_{Ws}^k [\underline{\sigma}_s \, d\mathbf{Z}_s]. \tag{11}$$

The random wealth impact upon marginal utility derives from k's portfolio weights. Given the market portfolio holding of (MT^k/T^M) and the futures holdings as in (6), we have

[11] Actually, Cox, Ingersoll and Ross [11] showed that the riskless rate equals minus the expected rate of change of marginal utility in an economy with identical individuals. However, their proof can be used to derive that result for any single individual k, and the fact that the riskless rate is the same for all implies then that the expected rates of change of all individuals' marginal utilities must be the same.

$$\frac{1}{J_W^k}\,\sigma_{J_W}^k\,dZ^k = \frac{1}{J_W^k}\left\{J_{WW}^k\left[\frac{MT^k}{T^M}\,\sigma_M\,dZ_M + \left(\mathbf{H}_s^k - \frac{T^k}{T^M}\,\mathbf{H}_s^M\right)\underline{\boldsymbol{\sigma}}_s\,dZ_s\right]\right.$$
$$\left. + \mathbf{J}_{Ws}^k\underline{\boldsymbol{\sigma}}_s\,dZ_s\right\}$$
$$= -\frac{M}{T^M}\,\sigma_M\,dZ_M + \frac{\mathbf{H}_s^M}{T^M}\,\underline{\boldsymbol{\sigma}}_s\,d\mathbf{Z}_s, \tag{12}$$

which is the same for all individuals k. This demonstrates that, with futures, all individuals' marginal utilities are locally perfectly correlated.

The remaining, second term of (10) is proportional to $(\sigma_{J_W}^k)^2/(J_W^k)^2$. Again, Ito's lemma can be used to find the variance of k's marginal utility as follows:

$$(\sigma_{J_W}^k)^2 = (J_{WW}^k\mathbf{J}_{Ws}^k)\begin{pmatrix} V_{WW}^k & \mathbf{V}_{Ws}^k \\ \mathbf{V}_{sW}^k & \underline{\mathbf{V}}_{ss}\end{pmatrix}\begin{pmatrix} J_{WW}^k \\ \mathbf{J}_{sW}^k\end{pmatrix}. \tag{13}$$

This can be computed by using k's portfolio weights to compute k's wealth variance, V_{WW}^k, and k's covariances of wealth with the state vector, $\mathbf{V}_{sW}^k$ and then substituting those into (13). The result is that the second term of (10) is proportional to

$$\frac{(\sigma_{J_W}^k)^2}{(J_W^k)^2} = \left(\frac{M}{T^M}\right)^2\sigma_M^2 + \frac{2M}{(T^M)^2}\,\mathbf{V}_{Ms}\mathbf{H}_s^M + \frac{1}{(T^M)^2}\,\mathbf{H}_s^M\underline{\mathbf{V}}_{ss}\mathbf{H}_s^M \tag{14}$$

which is the same for all individuals. Equation (14) could also be obtained straight from (12) and the variance-covariance matrix for (M, s).

Sufficiency has now been proven since all parts of the change in the log of marginal utility in (10) are the same for all individuals.

Consider individuals' portfolio demands, (5), when there are not portfolios that perfectly hedge against all state variables' changes: each individual's net hedging demands, $\mathbf{H}_s^k - (T^k/T^M)\,\mathbf{H}_s^M$ go to the portfolios $\underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as}$, which were shown by Breeden [7] to have maximum correlations with respect to state variables. Now reformulate the portfolio problem slightly, letting the first S assets be those portfolios $\underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as}$, and letting the remaining assets be those $A - S$ assets that, when combined with the hedge portfolios, span the same space as the original A assets. Clearly, the same returns are possible and optimal as before. However, letting $\underline{\boldsymbol{\sigma}}_p\,d\mathbf{Z}_p$ be the stochastic components of returns on the hedge portfolios, the stochastic component of k's marginal utility that corresponds to Eq. (11) is

$$\frac{1}{J_W^k}\,\sigma_{J_W}^k\,dZ^k = \frac{1}{J_W^k}\left\{J_{WW}^k\left[\frac{MT^k}{T^M}\,\sigma_M dZ_M + \left(\mathbf{H}_s^k - \frac{T^k}{T^M}\,\mathbf{H}_s^M\right)\underline{\boldsymbol{\sigma}}_p\,d\mathbf{Z}_p\right]\right.$$
$$\left. + \mathbf{J}_{Ws}^k\underline{\boldsymbol{\sigma}}_s\,d\mathbf{Z}_s\right\}$$
$$= \frac{-M}{T^M}\,\sigma_M\,dZ_M + \frac{\mathbf{H}_s^M}{T^M}\,\underline{\boldsymbol{\sigma}}_p\,d\mathbf{Z}_p - \frac{\mathbf{H}_s^k}{T^k}\,[\underline{\boldsymbol{\sigma}}_p\,d\mathbf{Z}_p - \underline{\boldsymbol{\sigma}}_s\,d\mathbf{Z}_s]. \tag{15}$$

If there are not perfect hedges available, $\boldsymbol{\sigma}_p\, d\mathbf{Z}_p \neq \boldsymbol{\sigma}_s\, d\mathbf{Z}_s$ for some states of the world. In those cases, preferences ($\mathbf{H}_s^k$) can be chosen so that individuals' marginal utilities are not perfectly correlated, which implies a non-optimal allocation. Q.E.D.

IV. Supply and Demand for Futures Contracts

In the previous section, Theorem 3 demonstrated that continuous trading in "futures contracts" for elements of the consumption and investment opportunity set allow individuals to achieve an unconstrained Pareto-optimal allocation of risk. The characteristics of Pareto-optimal allocations, as given earlier by Theorem 1 and Theorem 2, have not changed since a central planner is not concerned with whether trading is continuous, discrete, or once-and-for-all; the planner simply maximizes expected utilities subject to resource constraints. Thus, continuous trading in "unconditional" futures contracts achieves a sharing of commodity prices' risks and aggregate expenditure's risk that is not necessarily linear or additive in commodity prices and expenditure. For example, continuous trading in futures contracts can achieve the contingent payoff function at a future time t of $\tilde{e}_t = \tilde{p}_{1t}^{\alpha_1} \tilde{p}_{2t}^{\alpha_2}$, where buy-and-hold strategies of futures and commodity options could not. This construction utilizes the important insights of Black and Scholes [5] in their demonstration that the nonlinear payoffs of an option can be replicated by continuous trading in the underlying asset and a bond.

This section examines individuals' holdings of these futures contracts and relates them to individuals' preferences for the various consumption goods and to individuals' differential exposures to reinvestment risks. The analysis of this section generalizes and extends some of the results obtained by Merton [28], Long [26], Fischer [18], Dieffenbach [15], and Grauer and Litzenberger [19] in their examinations of the consumption and investment hedging aspects of individuals' portfolios. Since the continuous-time model is the same as in Breeden's [7] paper, which is a multi-good version of Merton's [28] model, the consumption-oriented asset pricing theory applies. Given that, the focus of this section is entirely upon portfolio theory and hedging.

Individual k's portfolio of futures contracts was shown in Section III to be

$$\mathbf{w}_s^k W^k = \mathbf{H}_s^k - \frac{T^k}{T^M} \mathbf{H}_s^M. \tag{6}$$

Thus, it is clear that whether in equilibrium an individual demands (is net long) a particular futures contract or supplies it (is net short) depends critically upon the magnitude of his element of $\mathbf{H}_s^k$ relative to his risk tolerance share of the aggregate vector, $\mathbf{H}_s^M$. To understand these supplies and demands for futures, an analysis of the $\mathbf{H}_s^k$ vector follows.

Applying the implicit function theorem to the function for marginal utility of wealth, $J_W^k(W^k, \mathbf{s}, t)$, gives that H_{sj}^k is the compensating variation in wealth for a change in state variable j that is required to maintain the current level of marginal utility of wealth, i.e.,

$$H_{sj}^k = \frac{-J_{Ws_j}^k}{J_{WW}^k} = \left.\frac{\partial W^k}{\partial s_j}\right|_{J_W}. \tag{16}$$

As noted, at the individual's optimum, his marginal utility of wealth must equal his marginal utility of nominal consumption expenditure:

$$J_W^k(W^k, \mathbf{s}, t) = u_e^k(e^k, \mathbf{P}, t). \tag{17}$$

An alternative calculation of H_{sj}^k is given by finding $J_{Ws_j}^k$ and J_{WW}^k by implicit differentiation of [17], then using the implicit function theorem for the expenditure function[12]

$$H_{sj}^k = \left.\frac{\partial W^k}{\partial s_j}\right|_{e^k} = \left.\frac{\partial W^k}{\partial s_j}\right|_{u^k}, \quad \text{for elements of the investment opportunity set,} \tag{18}$$

$$H_{sj}^k = \left(\left.\frac{\partial e^k}{\partial P_j}\right|_{u_e^k} - \frac{\partial e^k}{\partial P_j}\right) \Bigg/ \left(\frac{\partial e^k}{\partial W^k}\right), \quad \text{for consumption-goods prices, } P_j. \tag{19}$$

It is shown in Appendix B that the vector of *commodity* futures demand components, given by (19) and now denoted $\mathbf{H}_c^k$, may be rewritten in a more instructive form:

$$\begin{aligned}\mathbf{H}_c^k &= e^k \left(\boldsymbol{\alpha}_c^k - \frac{\partial \ln e^k}{\partial \ln \mathbf{P}}\right) \Bigg/ \left(\frac{\partial e^k}{\partial W^k}\right) - T^k \mathbf{m}^k \\ &= \left.\frac{\partial W^k}{\partial \mathbf{P}}\right|_{u^k} - T^k \mathbf{m}^k,\end{aligned} \tag{20}$$

where $\boldsymbol{\alpha}_c^k$ is individual k's current vector of budget shares spent on the various consumption-goods (his average propensities to consume), and $\mathbf{m}^k$

[12] To get (18) and (19), note that $e^k = e^k(W^k, \mathbf{s}, t)$. Given this, for elements of the investment opportunity set (i.e., not commodity prices in $\mathbf{P}$), differentiation of (17) with respect to s_j gives $J_{Ws_j}^k = u_{ee}^k e_{s_j}^k$. For elements of the consumption-goods price vector, $\mathbf{P}$, which are elements of the state vector $\mathbf{s}$, we have: $J_{WP_j}^k = u_{ee}^k e_{P_j}^k + u_{eP_j}^k$. Differentiating (17) with respect to wealth gives: $J_{WW}^k = u_{ee}^k e_W^k$. Substituting these expressions into definition $H_{sj} = -J_{Ws_j}^k / J_{WW}^k$ gives (18) and (19) for investment opportunity set elements and for consumption opportunity set elements, respectively.

are individual k's marginal propensities to consume, $\underline{\mathbf{P}}(\partial \mathbf{c}^k/\partial e^k)$. That is, $\mathbf{m}^k$ represents the fractions of an additional dollar's consumption expenditure that k would spend on the various commodities.

Two examples will be given to indicate the implications of this analysis for individuals' holdings of future contracts. First, consider the effect on an individual of an increase in an interest rate, *ceteris paribus*. Assume that the interest rate increase has a real wealth effect that is positive, thereby tending to increase current consumption expenditure, e. However, the change has a negative effect on current consumption, in that the price of current consumption has increased relative to the price of future consumption. The net result on current consumption is ambigious; thus, individual k's demand component H_{sj}^k may be either positive or negative. In any case, those who consume more with an increase in interest rates would tend to be long in bonds, and those who consume less would tend to be short in bonds. Since equilibrium prohibits all investors from being net short or net long, equilibrium expected excess returns must create asset demands such that markets are cleared; this effect explains the second component of futures demands in (6), $-(T^k/T^M)\mathbf{H}_s^M$.

The second example of an individual's futures market position is of an increase in a consumption-goods price, P_j. From (20), an individual who increases current total consumption expenditure would tend to be short in this commodity's futures market, whereas an individual who decreases current expenditures would tend to be long in this futures market. Furthermore, an individual with a very low income elasticity of demand for the given commodity would tend to be long in this futures market, as the good would have a relatively small share of his marginal consumption bundle. This has an intuitive basis in that goods that are "necessities" (low income elasticities of demand) are "hedged" more than "luxuries" (goods with high income elasticities).

A more complete analysis of individuals' supplies and demands for commodity futures contracts is given in Section V for the case where an invariant price index exists, i.e., when individuals have unitary income elasticities of demand for all goods. First, however, consideration will be given to the more general concept of hedging against changes in consumption *and* investment opportunities in this multi-period model, with particular emphasis upon the role of futures markets as hedging instruments.

Merton [28], in a single-good model with changing investment opportunities, has characterized the $\mathbf{H}_s^k$ demands as "hedging" demands. The sense in which this is an apt characterization should be examined. From (18), futures contracts for investment set state variables held due to $\mathbf{H}_s^k$ will provide state-contingent wealth that combines in effect with investment opportunity changes to precisely maintain the utility of current consumption. This is a myopic view of hedging, as only the stability of an individual's utility of *current* consumption is considered. Since an individual chooses a consumption bundle and asset portfolio to maximize his expected utility of *lifetime* consumption, a more appropriate concept of hedging considers the use of futures contracts in stabilizing an individual's expected utility of lifetime consumption.

A perfect hedge, as defined here, is a portfolio of assets whose return in the various states of the world is such that the individual's utility of lifetime consumption, $J^k(W^k, \mathbf{s}, t)$, is the same in all states of the world. Consequently, an individual's hedging portfolio would have weights that are the compensating variations in wealth required to maintain expected lifetime utility, i.e., the weights would be $(-\mathbf{J}_s^k/J_W^k)$.

To examine the relation of the portfolio demand component $\mathbf{H}_s^k$ (which is Merton's hedging portfolio) to the lifetime hedging portfolio defined here, the following simplifying assumption is made for the remainder of this section:

(A4) Individual k's vector of percentage compensating variations in wealth for changes in the state variables is not a function of k's wealth level.[13,14]

Mathematically, this assumption is that: $(\partial/\partial W^k)[\mathbf{J}_s^k/W^k J_W^k] = \mathbf{0}$, which implies that

$$\mathbf{H}_s^k = \frac{-\mathbf{J}_{sW}^k}{J_{WW}^k} = W^k(1 - T^{*k})\,\boldsymbol{\gamma}_s^k, \tag{21}$$

[13] Assumption A4 will be true with complete (or Pareto optimal) capital markets for individuals with time-additive isoelastic utility functions. This is easiest to see in a discrete-time, multi-period state preference model, where the individual's subjective probability belief today for the occurrence of time-state ts is π_{ts}, the individual's wealth is W_0, and the price of a \$1 claim for time-state ts is ϕ_{ts}. With those definitions, the individual maximizes the Lagrangian: $L = \sum_t \sum_{s \in S_t} \pi_{\text{ts}} \rho^t c_{\text{ts}}^{1-\gamma} + \lambda[W_0 - \sum_t \sum_s \phi_{\text{ts}} c_{\text{ts}}]$. First-order conditions are the budget constraint and E conditions of the form:

$$\pi_{\text{ts}} c_{\text{ts}}^{-\gamma} \rho^t = \phi_{\text{ts}} c_0^{-\gamma},$$

which may be rewritten as $c_{\text{ts}} = c_0(\phi_{\text{ts}}/\pi_{\text{ts}}\rho^t)^{-1/\gamma}$, where E is the number of time-states. In the latter form of the first-order conditions, it is seen that there are E linear equations in he $E+1$ unknowns, $[c_0, (c_{\text{ts}})]$. Combining these and the budget equations into a matrix system, we may write $\underline{\mathbf{A}}\mathbf{c} = W_0\mathbf{b}$, where $\underline{\mathbf{A}}$ is the coefficient matrix, $\mathbf{c}$ is the vector of contingent consumption claims purchased, and $\mathbf{b}$ is a vector with a one in the first position and zeroes elsewhere. Note that the coefficient matrix, $\underline{\mathbf{A}}$, depends on probability beliefs, contingent claim prices, relative risk aversion, and pure time preference, but not upon initial wealth. The consumption vector may be found by pre-multiplying by $\underline{\mathbf{A}}^{-1}$, i.e., $\mathbf{c} = (\underline{\mathbf{A}}^{-1}\mathbf{b})\,W_0$. Substituting these optimal consumption claim purchases back into the objective function, we find that the indirect utility function for wealth may be written as $J(W, \mathbf{s}, t) = W^{1-\gamma} f(\mathbf{s}, t)$, where $\mathbf{s}$ describes the state of the world and $\underline{\mathbf{A}} = \underline{\mathbf{A}}(\mathbf{s}, t)$. It is easily verified that this utility function satisfies assumption A4.

[14] A similar assumption was made by Dieffenbach [15, 16], who also provided some estimates for percentage compensating variations in wealth for changes in consumer prices and Treasury bill rates.

where $\boldsymbol{\gamma}_s^k$ is k's vector of percentage compensating variations, i.e., $\boldsymbol{\gamma}_s^k = -\mathbf{J}_s^k / W^k J_W^k$; and T^{*k} is k's relative risk tolerance, as defined earlier.

Given assumption (A4), individual k's portfolio of futures can be rewritten by substituting (21) into (6):

$$\mathbf{w}_s^k = (1 - T^{*k})(\boldsymbol{\gamma}^k - \boldsymbol{\gamma}^M) + \left(1 - \frac{T^{*k}}{T^{*M}}\right) \boldsymbol{\gamma}^M + T^{*k}(\boldsymbol{\gamma}_T^M - \boldsymbol{\gamma}^M), \qquad (22)$$

where $\boldsymbol{\gamma}^M = \sum_k (W^k/M) \boldsymbol{\gamma}^k$ and $\boldsymbol{\gamma}_T^M = \sum_k (T^k/T^M) \boldsymbol{\gamma}^k$ are wealth and risk tolerance-weighted averages of individual's compensating variations, respectively.

From (22), an individual will hedge, in the sense that he will hold futures long for state variables whose increases hurt him more than the average person, if his relative risk tolerance is less than unity (the logarithm). If he is more risk tolerant than the logarithm, then the individual will tend to "reverse hedge," being short in futures for state variables whose increases hurt him more than the "typical" person. This generalizes similar results obtained by Dieffenbach [15] and by Grauer and Litzenberger [19] to investment opportunities, as well as consumption opportunities, and to more general utility functions.[15]

Individuals who have greater relative risk tolerance than the average will bear more of the social risk of changes in consumption and investment opportunities. The final term in (22) is a residual portfolio that will be the same for all individuals; it represents the difference between the risk tolerance weighted average index weights and the wealth-weighted average. If all individuals have identical relative risk tolerance, then this term is a vector of zeroes. Of course, an individual who is infinitely risk averse (zero risk tolerance) would perfectly hedge against changes in investment and consumption opportunities, i.e., $\mathbf{w}_s^k = \boldsymbol{\gamma}^k$, and would hold no other risky assets.

Equations (4) and (5) give the optimal portfolio of all assets for individual k. From (5), the total optimal portfolio consists of market portfolio holdings, futures holdings (6), and holdings of the riskless asset. The futures components of individuals' optimal portfolios have been described and related to individuals' consumption bundles and to their compensating variations in wealth for state variables' changes. An individual's total portfolio (market, futures, and riskless asset) can be viewed as a mean-

[15] The "reverse hedging" possibility was discussed in a 1974 version of the Grauer–Litzenberger [19] paper. However, their model was quite different, being a 2-period state preference model with complete markets. Stochastic investment opportunities were not considered by them. Dieffenbach [15] considered hedging against changes in investment opportunities in a multi-period model and also found log utility to be the dividing line between hedging and reverse hedging.

variance efficient portfolio in his "real wealth," as shown by the following analysis.

An individual's asset portfolio is chosen to maximize his expected change in real wealth for a given variance of real wealth.[16] The individual's real wealth, W^*, is defined as nominal wealth deflated by an inverse index of consumption and investment opportunities, i.e., $W^{*k} = W^k/I^k(\mathbf{s}, W^k)$. An asset's portfolio weight is an increasing function of its contribution to the individual's expected change in real wealth, and it is a decreasing function of its contribution to the variance of real wealth. With a wealth-invariant wealth deflator, an individual's asset portfolio, (4) may be restated as

$$\begin{aligned} \mathbf{w}^k &= T^{*k}\underline{\mathbf{V}}_{aa}^{-1}(\boldsymbol{\mu} - r) + \underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{as}\boldsymbol{\gamma}^k(1 - T^{*k}) \\ &= \underline{\mathbf{V}}_{aa}^{-1}\underline{\mathbf{V}}_{aI_k} + T^{*k}\underline{\mathbf{V}}_{aa}^{-1}(\boldsymbol{\mu} - \mathbf{r} - \mathbf{V}_{aI_k}), \end{aligned} \tag{23}$$

where $\mathbf{V}_{aI_k}$ is the $A \times 1$ vector of covariances of assets' returns with the individual's wealth deflator. If futures contracts exist for all of the state variables for the opportunity set, then this simplifies to

$$\mathbf{w}^k = \begin{pmatrix} \boldsymbol{\gamma}^k \\ \mathbf{0} \end{pmatrix} + T^{*k}\left[\underline{\mathbf{V}}_{aa}^{-1}(\boldsymbol{\mu} - \mathbf{r}) - \begin{pmatrix} \boldsymbol{\gamma}^k \\ \mathbf{0} \end{pmatrix}\right]. \tag{24}$$

The individual's total portfolio demands with incomplete markets, (23), and those with effectively complete markets, (24), can be explained as follows. The portfolio given by $\underline{\mathbf{V}}_{aa}^{-1}\mathbf{V}_{aI_k}$ has the maximum correlation of return with the wealth deflator and, therefore, is the best hedge against shifts in the opportunity set. If futures contracts exist, that portfolio is a futures portfolio with weights $\boldsymbol{\gamma}^k$. These holdings may be viewed as a result of the minimization of the variance of real wealth, i.e., they represent normal hedging demands.

As an asset's real value is its nominal value divided by the wealth deflator, $P^* = \tilde{P}/\tilde{I}$, the expression $(\boldsymbol{\mu} - \mathbf{r} - \mathbf{V}_{aI_k})$ is the continuous-time vector of expected real returns on assets in excess of the expected real return on the nominally riskless asset. Assets that have positive covariances with the deflator have lower expected real returns. Thus, there are offsetting considerations in the holdings of assets to "hedge" against opportunity set change. Variance minimization considerations result in positive holdings of assets that hedge against opportunity set shifts, while mean maximization considerations result in offsetting negative holdings of the same assets. The net result is ambiguous, depending upon the individual's risk tolerance (as discussed). The reverse hedging possibility of the myopic hedging portfolio,

[16] See Breeden's 1977 Stanford University dissertation for a definition of real wealth with stochastic opportunities and for a proof of the efficiency result in this model.

$\mathbf{H}_s^k$, is due to its juxtaposition of the conflicting effects of the holdings on real wealth's mean and variance.

V. Cobb–Douglas Consumption Preferences: An Example

In this example, it is assumed that all consumers have utility functions for goods consumed that are members of the Cobb–Douglas class with homogeneity of unity. That is, $U(\mathbf{c}, t) = U(\prod_{j=1}^{c} c_j^{\alpha_j}, t)$, where $\sum_j \alpha_j = 1$.

The optimal budget share for any commodity j is the constant α_j^k, regardless of total expenditures, e^k, or prices, $\mathbf{P}$. Demand functions have unitary expenditure elasticities, zero cross-price elasticities, and unitary income elasticities. Budget shares, $\boldsymbol{\alpha}^k$, could vary non-stochastically over time without changing the nature of the results, thus allowing consumption preferences to vary with age. However, this complication will be avoided here.

It is well-known[17] that an expenditure-invariant price index exists for individuals with these homothetic preferences. The vector of budget shares gives the weights for the price index. If it is further assumed that the only changes in consumption and investment opportunities are changes in consumption-goods prices and that those prices follow a random walk over time, then the same price index would be a valid deflator for nominal wealth. That is, real wealth would simply be nominal wealth deflated by the price index. The vector of percentage compensating variations in wealth for percentage changes in consumption-goods prices would be the individual's vector of budget shares, $\boldsymbol{\alpha}^k$; thus, the individual's demands for commodity futures are as given in the general equation (22), but with $\boldsymbol{\gamma}^k$ identified as $\boldsymbol{\alpha}^k$.

The analysis of which investors will be long and which will be short proceeds much as in the previous section. To examine the effects of hedging and risk allocation on the supply and demand for futures, two polar cases are instructive: (1) assume that all individuals have the same level of relative risk tolerance, but different vectors of budget shares, or (2) assume that all individuals have the same vector of budget shares (or price index), but different levels of risk tolerance. Consider first the case where all investors have the same level of relative risk tolerance, and let it be T^*. From (22) and the above discussion, individual k's wealth shares in the various commodity futures contracts, $\mathbf{w}_c^k$, are

$$\mathbf{w}_c^k = (1 - T^*)(\boldsymbol{\alpha}^k - \boldsymbol{\alpha}^M). \tag{25}$$

Therefore, when relative risk tolerance is less than unity, if an individual consumes more of a good than the aggregate, then he will be long in that

[17] See Samuelson and Swamy [33].

good's commodity futures market. Unless the individual's consumption bundle is precisely equal to the average bundle, he will be long in all futures such as those, and short in futures of goods that he consumes less of than the average. Conversely, if T^* is greater than unity, then all individuals would reverse hedge, being long in goods that they consume relatively less of and being short in futures for goods of which they consume relatively large amounts. In light of the previous section's analysis, the reverse hedging possibility occurs when the increased real wealth variance effect is offset in the individual's expected utility by the increased expected change in real wealth created by the reverse hedging policy.

Consider now the other polar case–all investors have the same consumption bundles, $\boldsymbol{\alpha}^k = \boldsymbol{\alpha}$, but they have varying degrees of relative risk tolerance. In this case, investor k will hold the following portfolio of commodity futures.

$$\mathbf{w}_c^k = \boldsymbol{\alpha}\left(1 - \frac{T^{*k}}{T^{*M}}\right). \tag{26}$$

This shows that an investor will be long in *all* futures contracts if his relative risk tolerance is less than that of the market, and he will be short in all contracts if his risk tolerance is greater than the market. This result arises from the fact social risks and individual risks coincide for each individual in this case; such social risk and the rewards appropriate in equilibrium are allocated to individuals according to their respective tolerances for risk-bearing.

VI. Commodity Options

In Section II it was noted that, while buy-and-hold portfolios of commodity options spanned a much larger space than that spanned by forward contracts, commodity options did not span the space of efficient allocations. Section III demonstrated that continuous trading in futures contracts or forward contracts spanned the space of efficient allocations. Of course, continuous trading in commodity options can span the same space of efficient allocations, but why have options when continuous trading in futures will do the same job? The reason is that transactions costs and heterogeneous beliefs were not considered in the analysis of futures. Desired nonlinear payoffs may require less trading with commodity options than with futures or forwards, making commodity options cost-effective in non-linear risk allocation. This section briefly combines some of the results of the recent options literature with this paper's theorems to demonstrate that commodity options and options on portfolios of forward contracts have significant allocational roles in an economy with costly transactions.

TABLE I

Price of Beef	Payoffs on Call Options C(2)	C(3)	C(4)	Payoffs on Spreads: Spread A C(2)–C(3)	Spread B C(3)–C(4)	Butterfly Spreads Spread A–Spread B C(2)–2C(3)+C(4)
$1.00	0	0	0	0	0	0
2.00	0	0	0	0	0	0
3.00	1	0	0	1	0	1
4.00	2	1	0	1	1	0
5.00	3	2	1	1	1	0
6.00	4	3	2	1	1	0

Consider an economy where individuals have diverse beliefs about the future price of beef.[18] All agree that the price per pound at date t will be in the set {$1, $2, $3, $4, $5, $6}, but disagree on probabilities. The current futures price is $4. Individual k receives information that the price is more likely to be $3 or $5 than k previously believed, with $2 and $6 being less likely. It is not possible for k to profit from this new information by simply buying or selling a futures contract and, since k is therefore unlikely to trade on the information, the allocation will not be efficient and prices will not "fully reflect all information."[19]

The literature on option pricing has shown that commodity options on the price of beef will allow k to speculate precisely on the new information, making both the allocation and prices more efficient.[20] For readers unfamiliar with this construction, Table I shows the contingent payoffs on commodity call options with exercise prices of $2, $3, and $4, respectively, and shows how a $1 payoff contingent upon a $3 beef price is achieved by a portfolio ("butterfly spread") of those commodity options.[21] Since the method of construction is general, a set of commodity options with all exercise prices lets indivials create general risk sharing functions of the form $f(\tilde{P}_B)$. In the beef price example, individual k would sell $2 and $6 claims (portfolios of call options) and buy commodity option portfolios that pay when the price of beef is $3 or $5. Thus, marginal rates of substitution of beef in different states would be equated for all individuals with commodity options. Both prices and allocations should be more efficient as a result.

[18] Note that heterogeneous beliefs with respect to commodity prices are possible under assumption A3 and, hence, in Theorem 2.

[19] See Fama [17] for a discussion of the efficient market hypothesis.

[20] Hart's [24] result that the addition of a new market can make all worse off must be noted as a qualifier to this efficiency statement.

[21] The particular technique for constructing elementary claims in Table I was in Breeden and Litzenberger [8]. Equivalent techniques were presented by Ross [32] and by Banz and Miller [3].

Actually, the allocational efficiency problem is more complex than in the example. For allocational efficiency, the marginal rate of substitution of beef in one state of the world for beef in another state must be the same for all individuals. For example, if beef and pork are substitutes, then the marginal utility of beef increases with decreases in pork consumption. Trading in commodity options for beef will ensure that the *expected* marginal rates of substitution of beef in various states, *conditional* upon the price of beef, is the same for all individuals. The problem is that there may be a number of states of the world with the same price of beef, but different prices of pork. Commodity options for beef will have the same payoff in all states with the same price of beef; therefore, they will not span the space of potential payoff functions across these states. Portfolios of commodity options for pork will span states with different port prices, but will do so with payoffs that are independent of the price of beef, given the price of pork. Simple holdings of commodity options for beef and pork cannot span the general space of efficient allocations since the optimal holding today of beef options is dependent upon the (uncertain) future price of pork and since the optimal holding of pork options is dependent upon the future price of beef.

For general optimal allocations, claims must be available that pay \$1 only if the price of beef is \$3/pound and, simultaneously, the price of pork is \$2/pound. Such payoffs are particular mappings of commodity prices and of aggregate consumption into the real line and are, therefore, subsets of Ross's [32] multiple options. As was just argued, simple commodity options will not span that space of efficient allocations. However, Ross has shown that options on portfolios of the underlying assets (forward contracts) span the larger space that is spanned by multiple options. Combining Ross's result [32, Theorem 5] with Theorem 2 gives the following corollary to Theorem 2:

COROLLARY 1. *Given assumptions* A1 *and* A3, *for each date t there exists a portfolio of forward contracts (maturing at date t) on consumption goods prices and on aggregate nominal consumption, such that call options on the portfolio span the space of efficient allocations of consumption at date t.*

To see this result, assume that the prices of beef and pork may be any of {\$1, \$2, \$3, \$4, \$5, \$6}, so that there are 36 "states." A portfolio of 10 forward contracts for beef and 1 forward contract for pork has a payoff that is different in each state. Therefore, options on that portfolio of forward contracts will span all 36 states, giving general payoffs of the form $f(\tilde{P}_B, \tilde{P}_p)$. After the options on the portfolio are purchased, no additional trading is required until the options mature. At that time, k uses the proceeds to purchase his desired consumption bundle in the spot market.

Ross's [32] characterization of the portfolio in Corollary 1 as an "efficient fund" is apt in the sense that options on that fund's return span the space of efficient allocations. However, as Arditti and John [1] have shown, in general there are a virtual continuum of portfolios that span the same space and are, therefore, just as "efficient." The sets of weights that may be used for the portfolio of forward contracts depend upon the contingent price and expenditure combinations that are the possible states. As the commodity prices' joint probability distribution is made more fine (continuous in the limit), the weights in the portfolio of forwards for Corollary 1 involve more computations.

An alternative construction of a Pareto-optimal capital market in a multi-good economy will now be presented. It involves more complex financial instruments than Ross's simple options on a portfolio of forwards, but has simple and intuitive portfolio weights and spans the same space of allocations. The technique follows from the following observations: (1) As shown in Table I, a \$1 payment contingent upon a beef price of \$3/pound at time t can be obtained from a portfolio of commodity options for beef; let this be portfolio A. Similarly, let portfolio B be the portfolio of commodity options that pays \$1 if the price of pork is \$2/pound at time t. (2) The portfolio C that consists of portfolio A plus portfolio B has its maximum payoff when, simultaneously, beef is \$3/pound and pork is \$2/pound, and that maximum payoff equals \$2, the number of "contingencies." (3) An option on portfolio C, with an exercise price equal to the number of contingencies minus one, pays \$1 if and only if all contingencies are met, and pays zero otherwise. (4) By similar arguments, the same construction will work for a \$1 payoff contingent upon a number of commodity prices being equal to the vector $\mathbf{P}$. If there are N contingencies, then the exercise price of the option-on-a-portfolio-of-options is $N-1$.

This analysis demonstrates that if there exist sets of commodity options for a number of different goods, then a financial intermediary could buy those commodity options and issue warrants that would pay off only if commodity prices for the component goods simultaneously equaled pre-specified levels. With diverse preferences, each individual has a different price index; since the relation of such warrants' payoffs to the commodity price vector is precise, each individual could construct a personalized index bond from a complete set of those warrants. Potential hedging and speculative uses of these options on portfolios of commodity options are immediate.

In practice, the increased hedging and speculative precision that is attainable with such option portfolios is partially offset by the costs of setting up the portfolios. The more fine the partition, the lower the probability of a payoff and, hence, the lower the value of the payoff. With a number of tightly defined contingencies, each warrant would be nearly worthless. However, with the advent of widespread commodity options trading apparently on the horizon in the U.S., more sophisticated construction of payoffs contingent upon the prices of groups of goods is not implausible.

VII. Conclusion

This paper has described the roles of futures markets and commodity options in the optimal allocation of consumption across time-states. With certain assumptions, it was shown that contingent futures contracts are not necessary for allocational efficiency, since the same allocations can be attained by options on portfolios of commodity options or by continuous trading in unconditional futures contracts. In the continuous-time model, individuals' optimal holdings of the various futures contracts were derived and interpreted. Both normal hedging and reverse hedging are possible due to the offsetting effects of futures holdings on real wealth's mean and variance.

Appendix A

Proof of Theorem 1. Any Pareto-optimal allocation of time-state contingent vectors of consumption-goods solves: $\max \sum_k a^k U^k$ for a set of positive constants $\{a_k\}$, where the maximum is taken over all feasible allocations, which are subject to resource constraints. A central planner (competitive equilibrium with optimal capital markets) would maximize the Lagrangian:

$$\max_{\{\mathbf{c}_{ts}^k\}} L = \sum_k a^k \left[\sum_t \sum_{s\in S_t} \pi_{ts}^k u_t^k(\mathbf{c}_{ts}^k)\right] + \sum_t \sum_{s\in S_t} \boldsymbol{\lambda}_{ts}' \left(\mathbf{C}_{ts} - \sum_k \mathbf{c}_{ts}^k\right). \quad \text{(A1)}$$

With assumption A2, the RHS of (A1) can be rewritten as

$$\sum_t \sum_{\mathbf{C}_t} \sum_{s\in S_{t\mathbf{C}}} \pi_{ts|\mathbf{C}} \left[\sum_k a^k \bar{\pi}_{t\mathbf{C}}^k u_t^k(\mathbf{c}_{ts}^k) + \frac{\boldsymbol{\lambda}_{ts}'}{\pi_{ts|\mathbf{C}}} \left(\mathbf{C}_t - \sum_k \mathbf{c}_{ts}^k\right)\right], \quad \text{(A2)}$$

where $S_{t\mathbf{C}}$ is the set of all states at t with aggregate consumption vector $\mathbf{C}$. Thus, due to the assumption of time-additivity of utility functions, the central planner's problem may be solved separately for each time-state, *given* $\{a^k\}$ and $\{\mathbf{C}_{ts}'\}$. As shown by examining expression (A2), the assumptions of state independence of utility and of conditionally homogeneous beliefs as in A2 make the central planner's objective, resource constraint and, hence, solution the same for all states with the same aggregate consumption vector (assuming uniqueness of the solutions to these subproblems). The critical element in this proof is that the central planner's objective function and constraint are effectively the same for all states with the same vector of aggregate consumption. The only state-dependent element of the central planner's weights on $\{U_t^k\}$ is $\pi_{ts|\mathbf{C}}$, but this is assumed to be the same for all individuals and does not affect the maximum problem.

Necessity of A2, given A1, is easily seen from the first-order conditions to (A1), which imply if A2 is not assumed to hold:

$$\frac{\pi_{t\mathbf{C}}^k \pi_{ts_1|\mathbf{C}}^k u_{ts_1}'^k}{\pi_{t\mathbf{C}}^k \pi_{ts_2|\mathbf{C}}^k u_{ts_2}'^k} = \frac{\pi_{t\mathbf{C}}^j}{\pi_{t\mathbf{C}}^j} \frac{\pi_{ts_1|\mathbf{C}}^j}{\pi_{t\mathbf{C}_2|\mathbf{C}}^j} \frac{u_{ts_1}'^j}{u_{ts_2}'^j}, \qquad \forall s_1, s_2 \in S_{t\mathbf{C}} \tag{A3}$$

where $u_{ts_1}'^k$ is individual k's marginal utility in time state ts_1 of another unit of, say, good 1. Since $\mathbf{c}_{ts_1}^k = \mathbf{c}_{ts_2}^k$ implies that $(u_{ts_1}'^k / u_{ts_2}'^k) = 1$, for the theorem to hold it must be that

$$\left(\frac{\pi_{ts_1|\mathbf{C}}^k}{\pi_{ts_2|\mathbf{C}}^k}\right) = \left(\frac{\pi_{ts_1|\mathbf{C}}^j}{\pi_{ts_2|\mathbf{C}}^j}\right), \qquad \forall s_1, s_2 \in S_{t\mathbf{C}}.$$

Since $\sum_{s \in S_{t\mathbf{C}}} \pi_{ts|\mathbf{C}}^k = 1$ for all k, this condition implies A2. Q.E.D.

Proof of Theorem 2. At any given date, an individual k with state-independent consumption preferences has an optimal consumption bundle that depends upon nominal consumption expenditure, e_k, and consumption good prices, $\mathbf{P}$, at that date, i.e., $\mathbf{c}_{ts}^k = \mathbf{c}_{ts}^k(e_{ts}^k; \mathbf{P}_{ts})$. Any two states in which k has the same nominal expenditure and the same prices result in the same consumption bundle, as the solution to that subproblem is assumed to be unique. Consequently, to prove the first part of the theorem, it is sufficient to show that any two states at the same date with the same $(E, \mathbf{P})$ have the same optimal allocation of nominal expenditure, $\{e^k\}$, where $E = \sum_k e^k$.

Considering two states at data t with the same $(E, \mathbf{P})$, an optimal allocation of consumption expenditure results in the same marginal rate of substitution (mrs) of "dollars" in one state for dollars in the other state for all individuals, i.e.,

$$\begin{aligned} &\frac{\pi_{ts_1|\mathbf{E}} \pi_{t\mathbf{E}}^k u_e^k(e_{s_1}^k; \mathbf{P}, t)}{\pi_{ts_2|\mathbf{E}} \pi_{t\mathbf{E}}^k u_e^k(e_{s_2}^k; \mathbf{P}, t)} = \frac{\pi_{ts_1|\mathbf{E}} \pi_{t\mathbf{E}}^j u_e^j(e_{s_1}^j; \mathbf{P}, t)}{\pi_{ts_2|\mathbf{E}} \pi_{t\mathbf{E}}^j u_e^j(e_{s_2}^j; \mathbf{P}, t)} \\ &\Rightarrow \frac{u_e^k(e_{s_1}^k; \mathbf{P}, t)}{u_e^k(e_{s_2}^k; \mathbf{P}, t)} = \frac{u_e^j(e_{s_1}^j; \mathbf{P}, t)}{u_e^j(e_{s_2}^j; \mathbf{P}, t)}, \qquad \forall j, k \text{ and } \forall s_1, s_2 \in S_{t\mathbf{E}}, \end{aligned} \tag{A4}$$

where $S_{t\mathbf{E}}$ is a set of states with a common vector $(E, \mathbf{P})$. Since each individual has a utility function for nominal expenditure that is monotonic with $u_e > 0$ and $u_{ee} < 0$, $u_e^k(e_{s_1}^k; \mathbf{P}, t)/u_e^k(e_{s_2}^k; \mathbf{P}, t) > (=) (<) 1$ iff $e_{s_1}^k < (=) (>) e_{s_2}^k$. Thus, for (A4) to be satisfied, $e_{s_1}^k - e_{s_2}^k$ must be of the same sign for all individuals for all states s_1, $s_2 \in S_{t\mathbf{C}}$. However, this relation and the conservation equation cannot hold unless $e^k_{s_1} = e^k_{s_2}$, $\forall k$, $\forall s_1, s_2 \in S_{t\mathbf{C}}$. Part I is proven.

Necessity of A3 for the theorem, given A1, is proven in the same way it was proven for Theorem 1.

Appendix B

The definition of the consumer's indirect utility function is

$$u(e, t; \mathbf{P}_c) = \max_{\mathbf{P}'\mathbf{c}=e} U(\mathbf{c}, t) = \max_{\mathbf{c}}\{U(\mathbf{c}, t) + \lambda(e - \mathbf{P}_c'\mathbf{c})\}, \tag{B1}$$

and the first-order conditions for a maximum are

$$\mathbf{U}_c = \lambda \mathbf{P}_c, \qquad \mathbf{u}_p = -\lambda \mathbf{c}, \qquad \text{and the shadow price} \quad \lambda = u_e. \tag{B2}$$

It is most convenient to deal in changes in the logarithm of price changes. Doing so, from Eq. (19):

$$H_{\ln P_j} = \left(\frac{\partial e}{\partial \ln P_j}\bigg|_{u_e} - \frac{\partial e}{\partial \ln P_j} \right) \bigg/ \left(\frac{\partial e}{\partial W} \right). \tag{B3}$$

By differentiating the optimality conditions in (B2)

$$\frac{\partial e}{\partial \ln P_j}\bigg|_{u_e} = \frac{-P_j u_{p_j e}}{u_{ee}} = \frac{u_e}{u_{ee}} P_j \frac{\partial c_j}{\partial e} + P_j c_j, \tag{B4}$$

and

$$T = \frac{-J_W}{J_{WW}} = \frac{-(u_e/u_{ee})}{(\partial e/\partial W)}. \tag{B5}$$

By substituting (B4) and (B5) into (B3), one finds Eq. (20):

$$H_{\ln P_j} = e\left(\frac{P_j c_j}{e} - \frac{\partial \ln e}{\partial \ln P_j} \right) \bigg/ \left(\frac{\partial e}{\partial W} \right) - T P_j \frac{\partial c_j}{\partial e}. \tag{20}$$

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