Calculus BC - 2021 AP Live Review Session 5
Examining Differential Equations and Logistics

1. After completing the 2020 AP Exams, students uploaded their responses to be scored. Let E represent the total number of AP Calculus responses, in thousands, that have been uploaded t minutes after testing is complete. At time $t = 10$ minutes, 75 thousand responses have been uploaded and $E'(10) = 18$.

(**a**) Write an equation of the line tangent to the graph of E at $t = 10$. Use the tangent line to approximate $E(13)$.

$E'(10) = 18 \qquad E(10) = 75$

$E(t) - 75 = 18(t - 10)$

or

$E(13) = 75 + 18(3) = 129$ approximately 129,000 uploads were complted at $t = 13$ mins

(**b**) Yoangel believes that E can be modeled by the logistic equation $y = R(t)$ where $\frac{dR}{dt} = \frac{8}{25}R\left(1 - \frac{R}{300}\right)$ and $R(10) = 75$. According to this model, what is the rate, in thousands of responses per minute, at the moment when responses are being uploaded fastest?

From our DEQ, the carrying capacity, or maximum number of exam uploads, is 300.

The rate is a maximum when we rezch half the carrying capacity, or 150.

$$\frac{dR}{dt} = \frac{8}{25} \cdot 150\left(1 - \frac{150}{300}\right) = 48\left(\frac{1}{2}\right) = 24 \text{ thousand responses/minute}$$

(**c**) According to the model R in part (b), is the rate that responses are uploaded increasing or decreasing at time $t = 10$? Give a reason for your answer.

$$\frac{d^2R}{dt^2} = \frac{8}{25} \cdot \frac{dR}{dt}\left(1 - \frac{R}{300}\right) + \frac{8}{25}R\left(-\frac{1}{300} \cdot \frac{dR}{dt}\right)$$

$$= \frac{8}{25}\left(\frac{8}{25}R\left(1 - \frac{R}{300}\right)\right)\left(1 - \frac{R}{300}\right) + \frac{8}{25}R\left(-\frac{1}{300}\left(\frac{8}{25}R\left(1 - \frac{R}{300}\right)\right)\right)$$

$$\left.\frac{d^2R}{dt^2}\right|_{t=10} = \frac{8}{25}\left(\frac{8}{25}(75)\left(1 - \frac{75}{300}\right)\right)\left(1 - \frac{75}{300}\right) + \frac{8}{25}(75)\left(-\frac{1}{300}\left(\frac{8}{25}(75)\left(1 - \frac{75}{300}\right)\right)\right)$$

$$= \frac{8}{25}\left(24 \cdot \frac{3}{4}\right)\left(\frac{3}{4}\right) + 24\left(-\frac{1}{300}(24)\left(\frac{3}{4}\right)\right)$$

$$= \frac{8}{25} \cdot \frac{27}{2} + 24\left(-\frac{3}{50}\right) = \frac{108}{25} - \frac{36}{25} = \frac{72}{25}$$

The rate the responses are uploaded is increasing at $t = 10$ because $\left.\frac{d^2R}{dt^2}\right|_{t=10} > 0.$

Or more concisely,
We know that the logistic growth curve will be concave up prior to reaching $\frac{1}{2}L$, or 150. $R(10) = 75$ is located on that portion of the graph, therefore $\frac{d^2R}{dt^2} > 0$, thus the rate the responses are uploaded is increasing at $t = 10$ min

(**d**) For $0 \le t \le 30$ minutes, Emma believes that E can be modeled by the function $y = U(t)$, where $\frac{dU}{dt} = \frac{2}{25}U$ and $U(10) = 75$ (U is measured in thousands of responses) Find the particular solution $y = U(t)$. According to this model, how many responses have been uploaded at time $t = 13$?

$$\frac{1}{U}dU = \frac{2}{25}dt$$

$$\int \frac{1}{U}dU = \int \frac{2}{25}dt$$

$$\ln|U| = \frac{2}{25}t + C$$

At $(10, 75)$,

$$\ln|75| = \frac{2}{25}(10) + C$$

$$C = -\frac{4}{5} + \ln 75$$

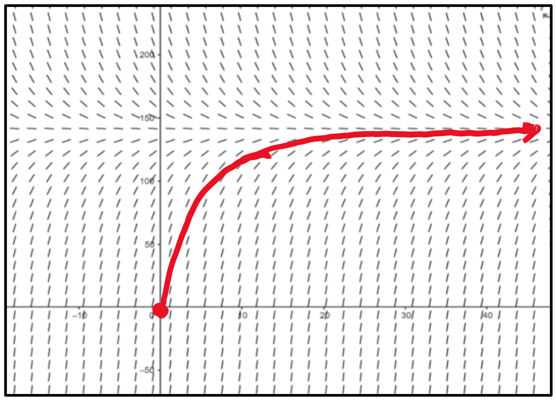
$$\ln|U| = \frac{2}{25}t - \frac{4}{5} + \ln 75$$

$$U = e^{\frac{2}{25}t - \frac{4}{5} + \ln 75} = 75e^{\frac{2}{25}t - \frac{4}{5}}$$

$$U(13) = 75e^{\frac{26}{25} - \frac{4}{5}} \text{ or } 75e^{\frac{6}{25}}$$

2. For $0 \le t \le 30$ minutes, the number of cars that have exited a parking garage can be modeled by the function $y = L(t)$ which satisfies the differential equation $\frac{dL}{dt} = \frac{2}{15}(280 - 2L)$. At time $t = 0$, there are 435 cars in the garage and no cars have exited the garage.

(**a**) Sketch the solution curve to the differential equation $\frac{dL}{dt} = \frac{2}{15}(280 - 2L)$ through the point $(0, 0)$.



(**b**) Use separation of variables to find the particular solution $y = L(t)$ to the differential equation above with condition $L(0) = 0$.

$$\int \frac{1}{280-2L}\,dL = \int \frac{2}{15}\,dt$$

$$-\frac{1}{2}\ln|280-2L| = \frac{2}{15}t + C$$

At $(0,0)$, $-\frac{1}{2}\ln(280) = C$

$$-\frac{1}{2}\ln|280-2L| = \frac{2}{15}t - \frac{1}{2}\ln(280)$$

$$\ln|280-2L| = -\frac{4}{15}t + \ln(280)$$

$$280-2L = e^{-\frac{4}{15}t+\ln(280)}$$

$$280-2L = 280e^{-\frac{4}{15}t}$$

$$-2L = 280e^{-\frac{4}{15}t} - 280$$

$$L(t) = 140 - 140e^{-\frac{4}{15}t}$$

(**c**) For $0 \le t \le 30$ minutes, cars enter the parking garage at a rate modeled by $E(t) = 6.7 - 1.2\cos\left(\frac{\pi t}{15}\right)$. Using your solution from part (b), find the number of cars in the parking garage at time $t = 15$.

Let $C(t)$ represent the number of cars in the parking garage at time t.

$$C(t) = 435 + \int_0^t E(x)\,dx - L(t)$$

$$C(15) = 435 + \int_0^{15} E(x)\,dx - L(15)$$

$$\approx 398.064$$

(**d**) For $0 \le t \le 30$ minutes, find the minimum number of cars in the parking garage. Justify your answer.

$$L'(t) = \frac{112}{3}e^{-\frac{4}{15}t}$$

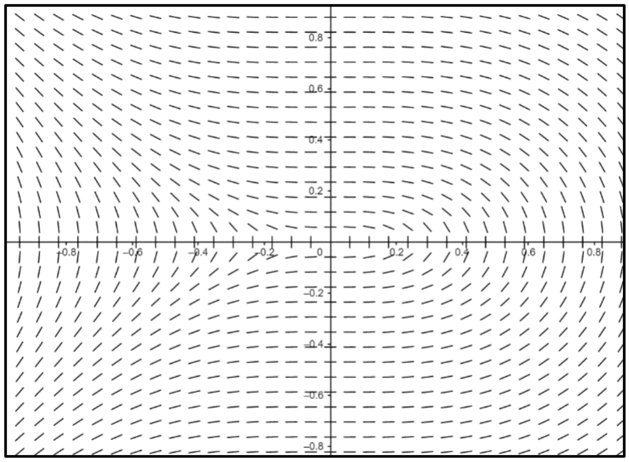
$L'(t) = E(t)$ when $t \approx 6.57348$

t	$C(t)$
0	435
6.57348	357.677
30	775.95

$$C(6.57348) \approx 435 + \int_0^{6.57348} E(t)\,dt - L(6.57348) \approx 357.677$$

$$C(30) \approx 435 + \int_0^{30} E(t)\,dt - L(30) \approx 775.95$$

The minimum number of cars in the garage is approximately 357.677 cars at time $t \approx 6.57348$ minutes.



1. Which of the following could be the equation of the differential equation that satisfies the slope field given in the figure above?

(A) $\frac{dy}{dx} = -\frac{x^2}{y}$ (B) $\frac{dy}{dx} = -\frac{x}{y}$ (C) $\frac{dy}{dx} = x - y$ (D) $\frac{dy}{dx} = \frac{y}{x}$

Testing the slope segment at $(1,1)$ which looks to be approximately -1

(A) $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{1} = -1$

(B) $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{1} = -1$

(C) $\left.\frac{dy}{dx}\right|_{(1,1)} = 1 - 1 = 0$ (Eliminated)

(D) $\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{1} = 1$ (Eliminated)

Testing the slope segment at $(-1,1)$ which looks to be approximately -1 as well

(A) $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{1}{1} = -1$

(B) $\left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{(-1)}{1} = 1$ (Eliminated)

2. Which of the following functions satisfies the differential equation $y'' - 2y' + 10 = 0$?

(A) $y = e^{2x} + 10$

(B) $y = e^{2x} + 10x$

(C) $y = 2e^{2x} + 5x$

(D) $y = 5e^{2x}$

(A) $y' = 2e^{2x};\ y'' = 4e^{2x} \rightarrow 4e^{2x} - 4e^{2x} + 10 \neq 0$

(B) $y' = 2e^{2x} + 10;\ y'' = 4e^{2x} \rightarrow 4e^{2x} - 4e^{2x} - 20 + 10 \neq 0$

(C) $y' = 2e^{2x} + 5;\ y'' = 4e^{2x} \rightarrow 4e^{2x} - 4e^{2x} - 10 + 10 = 0$

(D) $y' = 10e^{2x};\ y'' = 20e^{2x} \rightarrow 20e^{2x} - 20e^{2x} + 10 \neq 0$

3. A rumor is spreading that Trevor Packer is going to join Bryan and Tony for a guest appearance on AP Live. The total number of people that have heard the rumor R, in thousands, at time t hours follows a logistic model. Which of the following could be the logistic differential equation for the rate that the rumor is spreading at time t?

(A) $\frac{dR}{dt} = 10t\left(1 - \frac{t}{4000}\right)$

(B) $\frac{dR}{dt} = 10R\left(1 - \frac{t}{4000}\right)$

(C) $\frac{dR}{dt} = 4000t - 10t^2$

(D) $\frac{dR}{dt} = 4000R - 10R^2$

Neither choices (A), (B), nor (C) reflect the correct form of a logistic differential equation as the independent variable t is present on the right side.

Choice (D) can be rewritten as $\frac{dR}{dt} = 4000R\left(1 - \frac{R}{400}\right)$ which is a correct form for a logistic differential equation.

4. The function $y = f(x)$ satisfies the logistic differential equation $\frac{dy}{dx} = 6y\left(1 - \frac{y}{20}\right)$. It is known that $f(4) = 10$.

$\lim_{x \to 4} \frac{f(x) - 4x + 6}{f'(x) - 2x^2 + 2}$ is

(A) $-\frac{13}{8}$ (B) 0 (C) $\frac{96}{5}$ (D) nonexistent

$$\lim_{x \to 4}\left(f(x) - 4x + 6\right) = 10 - 16 + 6 = 0$$

$$\lim_{x \to 4}\left(f'(x) - 2x^2 + 6\right) = 60\left(1 - \frac{10}{20}\right) - 2(16) + 2 = 30 - 32 + 2 = 0$$

By L'Hospital's rule,

$$\lim_{x \to 4}\frac{f'(x) - 4}{f''(x) - 4x} = \frac{60\left(1 - \frac{10}{20}\right) - 4}{0 - 4(4)} = \frac{26}{-16} = -\frac{13}{8}$$

5. Let $y = H(t)$ be the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{10}(7 - 5H)$. Which of the following gives the value of $\frac{d^2H}{dt^2}$ at the point $H(2) = 3$?

(A) $-\frac{1}{2}$ (B) $\frac{2}{5}$ (C) $\frac{3}{2}$ (D) 4

$$\frac{d^2H}{dt^2} = \frac{1}{10}\left(-5 \cdot \frac{dH}{dt}\right)$$
$$= \frac{1}{10}\left(-5 \cdot \left(\frac{1}{10}(7 - 5H)\right)\right)$$
$$\left.\frac{d^2H}{dt^2}\right|_{H=3} = -\frac{1}{20}(7 - 5(3)) = \frac{8}{20} = \frac{2}{5}$$