AP CALCULUS BC **CHAPTER 9 REVIEW**

PLEASE REFRAIN FROM USING INSTRUMENTS OF WEAKNESS!

From the 2003 BC Exam:

1. The function *f* has a Taylor series about x = 2 that converges to $f(x)$ for all x in the interval of convergence. The *nth* derivative of *f* at x = 2 is given by the following

$f^{(n)}(2) = \frac{(n+1)!}{3^n}$ for $n \geq 1$ and $f(2) = 1$.

(a) Write the first four terms & the general term of the Taylor Series for *f* about x = 2.

(b) Find the radius of convergence for the Taylor Series for *f* about x = 2. **Show the work that leads to your answer!**

(c) Let *g* be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor Series for *g* about x = 2.

(d) Does the Taylor Series for *g* as defined in part (c) converge at x = -2? **Give a reason for your answer!**

Definitions of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c.** Moreover, if $c = 0$, then the series is the **Maclaurin series for f.**

a) $f(2) = 1$

$f'(2) = \frac{2}{3}$

$f''(2) = \frac{2}{3}$

$f'''(2) = 8/9$

$$\boxed{1 + \frac{2}{3}(x-2) + \frac{1}{3}(x-2)^2 + \frac{4}{27}(x-2)^3}$$

$$\sum_{n=0}^{\infty} \frac{(n+1)!}{3^n n!}(x-2)^n = \boxed{\sum_{n=0}^{\infty} \frac{n+1}{3^n}(x-2)^n}$$

b)

$$\lim_{n\to\infty}\left|\frac{\frac{n+2}{3^{n+1}}(x-2)^{n+1}}{\frac{n+1}{3^n}(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{3(n+1)}(x-2)\right| =$$

$$= \underbrace{\frac{1}{3}|x-2| < 1}_{|x-2|<3} \Rightarrow \boxed{R=3}$$

$$(-1, 5)$$

at $x=-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{3^n}(-3)^n = \sum_{n=0}^{\infty}(n+1)(-1)^n$ Diverge

at $x=1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)}{3^n}3^n = \sum_{n=0}^{\infty}n+1$ Diverge

Interval of Converge: $(-1,5)$

(c) Let g be a function satisfying $g(2)=3$ and $g'(x)=f(x)$ for all x. Write the first four terms and the general term of the Taylor Series for g about x = 2.

$g(2)=3 \quad g'(2)=f(2)=1$

$$f^{(n)}(2)=\frac{(n+1)!}{3^n}$$

$g''(2)=f'(2)=\frac{2}{3}$

$\boxed{3+(x-2)+\frac{1}{3}(x-2)^2+\frac{1}{9}(x-2)^3}$

$g'''(2)=f''(2)=\frac{2}{3}$

(d) Does the Taylor Series for g as defined in part (c) converge at $x = -2$?
Give a reason for your answer!

$x=-2$ is not in the interval of convergence for $f(x)$ and $g'(x)=f(x)$, therefore $x=-2$ is not in the interval of convergence of $g(x)$.

($f(x)$, $f'(x)$, and $\int f(x)dx = g(x)$ have the same int. of convergence (they may only differ at the endpoints.)

From the 1999 BC Exam:

2. Let *f* be a function that has derivatives of all orders for all real numbers. Assume $f(0)=5,\ f'(0)=-3,\ f''(0)=1,\ f'''(0)=4$

a) Write the third degree Taylor Polynomial for *f* about $x=0$ and use it to approximate $f(0.2)$.

b) Write the fourth degree Taylor Polynomial for g, where $g(x)=f(x^2)$, about $x=0$.

c) Write the third degree Taylor Polynomial for *h*, where $h(x)=\int_0^x f(t)dt$, about $x=0$.

d) Let *h* defined as in part c). Given that $f(1)=3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

a) $P_3(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{6}x^3$

$\boxed{P_3(x)=5-3x+\frac{1}{2}x^2+\frac{2}{3}x^3}$

$\boxed{f(0.2)\approx P_3(0.2)=4.425}$

b) $g(x)=f(x^2)$ For $f(x)$: $P_2(x)=5-3x+\frac{1}{2}x^2$

For $g(x)$: $P_4(x)=5-3(x^2)+\frac{1}{2}(x^2)^2=5-3x^2+\frac{1}{2}x^4$

$\boxed{P_4(x)=5-3x^2+\frac{1}{2}x^4}$

c) $h(x)=\int_0^x f(t)dt=\int_0^x 5-3t+\frac{1}{2}t^2\,dt=5t-\frac{3t^2}{2}+\frac{1}{6}t^3\Big]_0^x$

$=5x-\frac{3}{2}x^2+\frac{1}{6}x^3$

For $h(x)$: $\boxed{P_3(x) = 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3}$

d) $h(1) = \int_0^1 f(t)\,dt$ but we only know $f(t)$ at $t=0$ and $t=1$,

therefore we can not exactly know the $\int_0^1 f(t)\,dt$.

(We could use the poly. app. but that will be a diff. question)

3. a. Find a power series for $f(x)=\dfrac{3}{x+2}$ centered at 0.

b. Find the radius & interval of convergence for the above series.

a) $f(0)=3/2 \qquad f(x)=3(x+2)^{-1}$

$f'(x)=-3(x+2)^{-2}=-\dfrac{3}{(x+2)^2} \qquad f'(0)=-\dfrac{3}{4}=-3\cdot\dfrac{1}{2^2}$

$f''(x)=6(x+2)^{-3}=\dfrac{6}{(x+2)^3} \qquad f''(0)=\dfrac{6}{8}=3\cdot\dfrac{2}{2^3}$

$f'''(0)=-18(x+2)^{-4}=-\dfrac{18}{(x+2)^4} \qquad f'''(0)=-3\cdot\dfrac{6}{2^4}$

$f^{(n)}=(-1)^n\cdot\dfrac{3n!}{2^{n+1}}$

$$\boxed{\sum_{n=0}^{\infty}(-1)^n\frac{3}{2^{n+1}}x^n}$$

b) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{3}{2^{n+2}}x^{n+1}}{(-1)^n\frac{3}{2^{n+1}}x^n}\right|=\lim_{n\to\infty}\left|\frac{(-1)}{2}x\right|=\frac{|x|}{2}<1$$

$|x|<2$

$\boxed{R=2} \qquad (-2,2)$

at $x=-2$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3}{2^{n+1}}(-2)^n=\sum_{n=0}^{\infty}(-1)^{2n}\cdot\frac{3\cdot 2^n}{2^{n+1}}$

$\displaystyle\sum_{n=0}^{\infty}3/2 \quad \boxed{\text{Diverge}}$

at $x=2$: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3\,(2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} (-1)^n \frac{3}{2}$ $\boxed{\text{Diverge}}$

$\boxed{\text{Interval of Convergence: } (-2,2)}$

4. a. Build a MacLaurin expansion for $f(x) = \sin(\sqrt{x})$
b. For what values of x does this series converge?
c. Use your series from problem 4 to determine how many terms are needed to approximate $f(1.5)$ with error less than 10^{-6}

a) $g(x) = \sin x \quad g(0) = 0$

$g'(x) = \cos x \quad g'(0) = 1$

$g''(x) = -\sin x \quad g''(0) = 0$

$g'''(x) = -\cos x \quad g'''(0) = -1$

$0 + x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$f: \sum_{n=0}^{\infty} (-1)^n \frac{(x^{1/2})^{2n+1}}{(2n+1)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1/2}}{(2n+1)!}}$$

b) $$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{n+3/2}}{(2n+3)(2n+2)\cancel{(2n+1)!}}}{\frac{(-1)^n x^{n+1/2}}{\cancel{(2n+1)!}}} \right| = \lim_{n\to\infty} \left| \frac{(-1)\, x}{(2n+3)(2n+2)} \right| = 0 \ <1$$

$\boxed{R = \infty}$ $\boxed{(-\infty, \infty)}$ ← The series converge for all $\mathbb{R}$.

c) $$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \le \frac{x^{n+1}}{(n+1)!}$$

$$\frac{(1.5)^{n+1}}{(n+1)!} \le 10^{-6}$$ $\boxed{n = 12}$

```
       86.49755859
Ans/(11!)
    2.166946213E-6
1.5^12
       129.7463379
Ans/12!
    2.708682766E-7
```

5. Determine whether the following converge or diverge **– state what test you're using and find an upper bound if possible!**

a. $\sum_{n=o}^{\infty}\frac{2^n}{3^n+2}$

$a_n = \frac{2^n}{3^n+2} \le b_n = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$

$\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ Converge (Geo. Series w/ $r = 2/3$) $\Rightarrow$ $\boxed{\text{By the Direct Comparison Test } \sum_{n=0}^{\infty}\frac{2^n}{3^n+2} \text{ abs. converge}}$

$\frac{1}{1-2/3} = \frac{1}{1/3} = 3$

$\boxed{\sum_{n=0}^{\infty}\frac{2^n}{3^n+2} \le 3}$

b. $\sum_{n=1}^{\infty}\frac{n^{3/2}+5}{n^2+\sqrt{n}} \ge \sum_{n=1}^{\infty}\frac{n^{3/2}}{n^2+n^2} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{n^{3/2}}{n^2} = \frac{1}{2}\underbrace{\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}}_{\text{P-Series, } p=1/2 \text{ Diverge}}$

By Direct Comparison Test

$\sum_{n=1}^{\infty}\frac{n^{3/2}+5}{n^2+\sqrt{n}}$ Diverge.

c. $\sum_{n=1}^{\infty}\frac{\ln n^2}{n} = 2\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$

$\underbrace{\int_1^{\infty}\frac{\ln x}{x}\,dx = \int_0^{\infty} u\,du = \frac{u^2}{2}\Big]_0^{\infty} = \infty}$

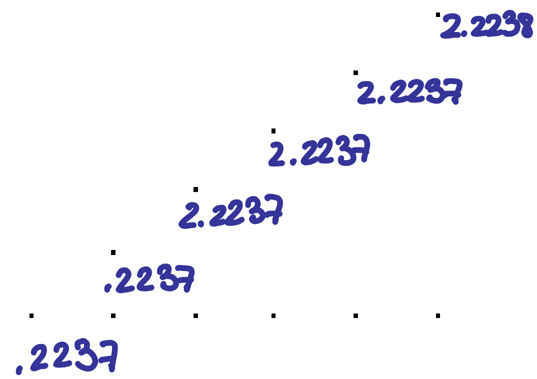
$\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ Diverge by the Integral Test

6. Consider the sequence $\left\{ \left(1+\frac{1}{5n}\right)^{4n} \right\}$

a. Graph using $95 \le n \le 100$. Indicate the coordinate of $a_{95}..a_{100}$ rounded to four places

b. Notice your sequence seems to converge? Find the actual value the sequence converges to using analytic means and show work!

a) Set the **MODE** to Seq in your Calculator



b) $\left\{ \left(1+\frac{1}{5n}\right)^{4n} \right\}$ $\quad \lim\limits_{n\to\infty} \left(1+\frac{1}{5n}\right)^{4n} = L$

$$\ln(L) = \lim_{n\to\infty} 4n \ln\left(1+\frac{1}{5n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{5n}\right)}{\frac{1}{4n}}$$

$$\ln(L) = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{5n}}\left(-\frac{1}{5\cancel{n^2}}\right)}{-\frac{1}{4\cancel{n^2}}} = \frac{4}{5}\lim_{n\to\infty} \frac{1}{1+1/5n} = 4/5$$

$$\ln(L) = 4/5$$

$$\boxed{L = e^{4/5}}$$

7. Recall the formal definition of a convergent sequence a.k.a. $\{a_n\} \to L$

For any $\varepsilon > 0$, $\exists\ M(\varepsilon) \ni$ if $n > M(\varepsilon)$ then $|a_n - L| < \varepsilon$

Find such a $M(\varepsilon)$ for the sequence $\left\{\dfrac{3n^2+1}{2n^2}\right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2+1}{2n^2} = 3/2$$

$$\left| a_n - 3/2 \right| < \varepsilon$$

$$\left| \frac{3n^2+1}{2n^2} - \frac{3n^2}{2n^2} \right| < \varepsilon$$

$$\left| \frac{1}{2n^2} \right| < \varepsilon$$

$$\frac{1}{2n^2} < \varepsilon$$

$$\frac{1}{n^2} < 2\varepsilon$$

$$n^2 > \frac{1}{2\varepsilon}$$

$$n > \frac{1}{\sqrt{2\varepsilon}}$$

$$\boxed{M(\varepsilon) = \frac{1}{\sqrt{\varepsilon}}}$$

3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.

(a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

(b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.

(c) Write the third-degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.

(d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $P_3(f)(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$

$f(0.2) \approx P_3(f)(0.2) =$

$5 - 3(0.2) + \frac{0.04}{2} + \frac{2(0.008)}{3} =$

4.425

$3 \begin{cases} 2\text{: } 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3 \\ \quad \langle -1 \rangle \text{ each incorrect term, extra term, or } + \cdots \\ 1\text{: approximates } f(0.2) \end{cases}$

$\langle -1 \rangle$ for incorrect use of $=$

(b) $P_4(g)(x) = P_2(f)(x^2) = 5 - 3x^2 + \frac{1}{2}x^4$

2: $P_2(f)(x^2)$

$\langle -1 \rangle$ each incorrect or extra term

(c) $P_3(h)(x) = \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$

$= \left[5t - \frac{3}{2}t^2 + \frac{1}{6}t^3\right]_0^x$

$= 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$

$2 \begin{cases} 1\text{: } P_3(h)(x) = \int_0^x P_2(f)(t)\,dt \\ 1\text{: answer} \\ \quad 0/1 \text{ if any incorrect or extra terms} \end{cases}$

(d) $h(1) = \int_0^1 f(t)\,dt$

cannot be determined because $f(t)$ is known only for $t = 0$ and $t = 1$

$2 \begin{cases} 1\text{: } h(1) \text{ cannot be determined} \\ 1\text{: reason} \end{cases}$

8. Which of the following statements about series is true?

B

~~(A)~~ If $\lim_{n\to\infty} u_n = 0$ then $\sum u_n$ converges.

(B) If $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ diverges.

~~(C)~~ If $\sum u_n$ diverges then $\lim_{n\to\infty} u_n \neq 0$.

~~(D)~~ (A) & (B)

~~(E)~~ (B) & (C)

9. Which of the following converges **absolutely**?

E

~~(A)~~ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$

~~(B)~~ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

~~(C)~~ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

~~(D)~~ $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

(E) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

10. Which of the following statements about series is false?

C

~~(A)~~ If $\sum u_n$ converges, so does $\sum cu_n$ if $c \neq 0$

~~(B)~~ If $\sum a_n$ and $\sum b_n$ converge, so does $\sum (ca_n + b_n)$, where $c \neq 0$

(C) If $\sum u_n$ converges, so does $\sum |u_n|$

~~(D)~~ If $\sum |u_n|$ converges, so does $\sum u_n$

~~(E)~~ Rearranging the terms of a positive convergent series will not affect its convergence or its sum

11. Which of the following converge?

E

(A) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(B) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

(C) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$

(D) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{\sqrt{n}}\right)$

(E) None of the above

$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = 1$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge ~~A~~

~~B~~

$\lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = 1$ and

$\lim_{n\to\infty} \frac{\sin(1/\sqrt{n})}{1/\sqrt{n}} = 1$ ~~C~~

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverge and

$\lim_{n\to\infty} \frac{\tan(1/\sqrt{n})}{1/\sqrt{n}} = 1$ ~~D~~

12. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sqrt{n+1}-\sqrt{n}\right)$

B

~~(A)~~ converges absolutely (B) converges conditionally

(C) diverges to ∞ (D) diverges to $-\infty$ (E) None of the above

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\left(\sqrt{n+1}-\sqrt{n}\right)\right| = \sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\underbrace{\lim_{n\to\infty}\left(\frac{\frac{1}{\sqrt{n+1}+\sqrt{n}}}{\frac{1}{2\sqrt{n}}}\right) = \lim_{n\to\infty}\frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = 1 \text{ and } \underbrace{\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}}_{\text{Diverge, P-Series, } p=1/2} \text{ By the Limit Comp. Test } \sum_{n=1}^{\infty}\left|(-1)^{n-1}\left(\sqrt{n+1}-\sqrt{n}\right)\right| \text{ Diverge.}}_{\cancel{A}}$$

B) $\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}=0$

$\sqrt{n+2}-\sqrt{n+1} \le \sqrt{n+1}-\sqrt{n}$

$\frac{1}{\sqrt{n+2}+\sqrt{n+1}} \le \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$a_{n+1} \le a_n$

By the Alt. Series Test $\sum_{n=0}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$ conditionally converges.

13. The Alternating Series Remainder Theorem can be applied to each of the following EXCEPT:

(A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$ (B) $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n+1}$ (C) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

E

(D) $\sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{\sqrt{n+1}}$ (E) $\sum_{n=0}^{\infty} \frac{\cos(2n\pi)}{n+1}$

always even

$\cos(2n\pi) = 1$

$\sum_{n=0}^{\infty} \frac{1}{n+1}$ not Alternating