

# Chapter 1: Complex Numbers (Year 2)

1 :: Exponential form of a complex number

$$\cos 2\theta + i \sin 2\theta \\ \rightarrow e^{2i\theta}$$

2 :: Multiplying and dividing complex numbers

If  $z_1$  and  $z_2$  are two complex numbers, what happens to their moduli when we find  $z_1 z_2$ . What happens to their arguments when we find  $\frac{z_1}{z_2}$ ?

3 :: De Moivre's Theorem

$$\text{If } z = r(\cos \theta + i \sin \theta), \\ z^n = r^n(\cos n\theta + i \sin n\theta)$$

4 :: De Moivre's for Trigonometric Identities

"Express  $\cos 3\theta$  in terms of powers of  $\cos \theta$ "

5 :: Roots

"Solve  $z^4 = 3 + 2i$ "

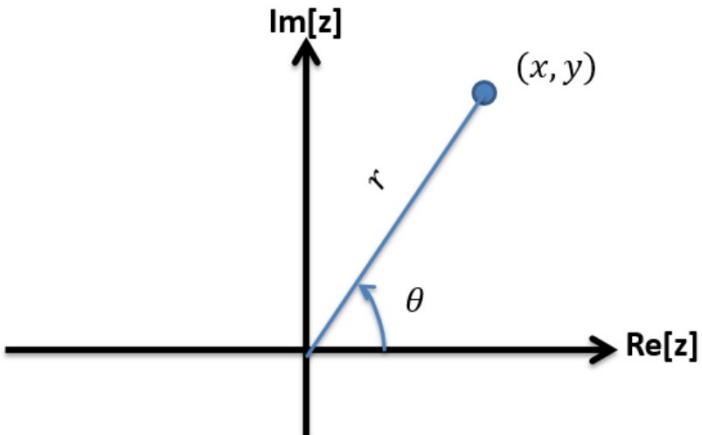
6 :: Sums of series

"Given that  $z = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ , where  $n$  is a positive integer, show that

$$1 + z + z^2 + \dots + z^{n-1} \\ = 1 + i \cot\left(\frac{\pi}{2n}\right)$$

"

## RECAP :: Modulus-Argument Form



If  $z = x + iy$  (and suppose in this case  $z$  is in the first quadrant), what was:

$$r = |z| = \sqrt{x^2 + y^2} \\ \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

When  $-\pi < \theta < \pi$  it is known as the **principal argument**.

Then in terms of  $r$  and  $\theta$ :

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= x + iy \\ &= r \cos \theta + i r \sin \theta \\ &= r(\cos \theta + i \sin \theta) \end{aligned}$$

This is known as the **modulus-argument form** of  $z$ .

$\sqrt{x^2 + y^2} \rightarrow \tan(\frac{y}{x})$  Then check which quadrant.

$x + iy$	$r$	$\theta$	Modulus-argument form
-1	1	$\pi$	$z = \cos \pi + i \sin \pi$
$i$	1	$\frac{\pi}{2}$	$z = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$
$1 + i$	$\sqrt{2}$	$\frac{\pi}{4}$	$z = \sqrt{2} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$
$-\sqrt{3} + i$	2	$\frac{5\pi}{6}$	$z = 2 (\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$

## Exponential Form

We've seen the Cartesian form a complex number  $z = x + yi$  and the modulus-argument form  $z = r(\cos \theta + i \sin \theta)$ . But, there's a third form!

Later in Chapter 3 on Taylor expansions, you'll see that you can write functions as an infinitely long polynomial:

$$\begin{aligned}\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots\end{aligned}$$

It looks like the  $\cos x$  and  $\sin x$  somehow add to give  $e^x$ . The one problem is that the signs don't quite match up. But  $i$  changes sign as we raise it to higher powers.

$$\begin{aligned}e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right) \\ &= \cos \theta + i \sin \theta\end{aligned}$$



## Exponential form

$$z = re^{i\theta}$$

You need to be able to convert to and from exponential form.

$x + iy$	Mod-arg form	Exp Form
-1	$z = \cos \pi + i \sin \pi$	$z = e^{i\pi}$
$2 - 3i$		$z = \sqrt{13} e^{-0.983i}$
	$\sqrt{2} \left( \cos \frac{\pi}{10} + i \sin \frac{\pi}{10} \right)$	$z = \sqrt{2} e^{\frac{\pi i}{10}}$
-1+i	$\sqrt{2} \left( \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$	$z = \sqrt{2} e^{\frac{3\pi i}{4}}$
	$2 \left( \cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right)$	$z = 2 e^{\frac{23\pi i}{5}}$

$$e^{i\pi} + 1 = 0$$

This is Euler's identity.  
It relates the five most fundamental constants in maths!

To get Cartesian form, put in modulus-argument form first.

Notice this is not a principal argument.

Use  $e^{i\theta} = \cos \theta + i \sin \theta$  to show that  $\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta$$

$$e^{i\theta} + e^{-i\theta} = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta$$

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta$$

$$\frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \cos \theta$$

# Multiplying and Dividing Complex Numbers

If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$   
Then:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

i.e. If you multiply two complex numbers, you **multiply the moduli** and **add the arguments**, and if you divide them, you divide the moduli and subtract the arguments.

Similarly if  $z_1 = r_1 e^{i\theta_1}$  and  $z_2 = r_2 e^{i\theta_2}$

Then:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$3 \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right) \times 4 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$12 \left( \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

$$12(0 + i(1)) = 12i$$

Write in the form  $re^{i\theta}$ :

$$\frac{2 \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)}{\sqrt{2} \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)} = \sqrt{2} \left( \cos -\frac{3\pi}{4} + i \sin -\frac{3\pi}{4} \right)$$
$$= \sqrt{2} e^{-\frac{3\pi}{4}i}$$

$$2 \left( \cos \frac{\pi}{15} + i \sin \frac{\pi}{15} \right) \times 3 \left( \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5} \right)$$

$\cos \theta - i \sin \theta$  can be written as  
 $\cos(-\theta) + i \sin(-\theta)$

$$\hookrightarrow 3 \left( \cos -\frac{2\pi}{5} + i \sin -\frac{2\pi}{5} \right)$$

$$2 \left( \cos \frac{\pi}{15} + i \sin \frac{\pi}{15} \right) \times 3 \left( \cos -\frac{2\pi}{5} + i \sin -\frac{2\pi}{5} \right)$$

$$6 \left( \cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3} \right)$$

$$3 - 3\sqrt{3}i$$

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$$z = 5\sqrt{3} - 5i$$

Find

(a)  $|z|$

(b)  $\arg(z)$  in terms of  $\pi$

(1)

(2)

$$w = 2 \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

Find

(c)  $\left| \frac{w}{z} \right|$

(d)  $\arg \left| \frac{w}{z} \right|$

(1)

(2)

$$a) \sqrt{(5\sqrt{3})^2 + (5)^2} = 10$$

$$b) -\tan^{-1} \left( \frac{5}{5\sqrt{3}} \right) = -\frac{\pi}{6}$$

$$c) \left| \frac{w}{z} \right| = \frac{2}{10} = \frac{1}{5}$$

$$d) \frac{\pi}{4} - \left( -\frac{\pi}{6} \right) = \frac{5\pi}{12}$$

10	$\arg(z) = \tan^{-1}(\text{Im } z / \text{Re } z)$	10
10	$\arg(w) = \tan^{-1} \left( \frac{\text{Im } w}{\text{Re } w} \right) = \frac{\pi}{4}$	10
10	$\frac{2}{5} \left( \frac{1}{2} + \frac{1}{2}i \right) = \frac{1}{5} + \frac{1}{5}i$	10
10	$\arg \left( \frac{1}{5} + \frac{1}{5}i \right) = \tan^{-1} \left( \frac{1}{5} / \frac{1}{5} \right) = \frac{\pi}{4}$	10



# De Moivre's Theorem

We saw that:  $z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$

Can you think therefore what  $z^n$  is going to be?

If  $z = r(\cos \theta + i \sin \theta)$   
 $z^n = r^n(\cos n\theta + i \sin n\theta)$

This is known as **De Moivre's Theorem**.

How about  $(z = re^{i\theta}) \Rightarrow z^n ?$   
 $z^n = r^n e^{in\theta}$

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Prove by induction that  $z^n = r^n(\cos n\theta + i \sin n\theta)$

$$z^k = r^k(\cos k\theta + i \sin k\theta)$$

$$z^{k+1} = z^k \times z$$

$$= r^k(\cos k\theta + i \sin k\theta) \times r(\cos \theta + i \sin \theta)$$

$$= r^k r(\cos(k\theta + \theta) + i \sin(k\theta + \theta))$$

$$= r^{k+1}(\cos((k+1)\theta + i \sin((k+1)\theta))$$

## De Moivre's Theorem for Exponential Form

If  $z = re^{i\theta}$  then  $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

Alternative: Using Euler's form

$$\begin{aligned} z &= r(\cos \theta + i \sin \theta) = re^{i\theta} \\ z^{k+1} &= z^k \times z = (re^{i\theta})^k \times re^{i\theta} = r^k e^{ik\theta} \times re^{i\theta} \\ &= r^{k+1} e^{i(k+1)\theta} \\ &= r^{k+1}(\cos((k+1)\theta + i \sin((k+1)\theta)) \\ k = 1 \quad z^1 &= r^1(\cos \theta + i \sin \theta) \\ \text{True for } n = 1 \therefore \text{true for all } n \text{ etc} \end{aligned}$$

$$\text{Simplify } \frac{\left(\cos \frac{9\pi}{17} + i \sin \frac{9\pi}{17}\right)^5}{\left(\cos \frac{2\pi}{17} - i \sin \frac{2\pi}{17}\right)^3} = \frac{\left(\cos \frac{45\pi}{17} + i \sin \frac{45\pi}{17}\right)}{\left(\cos -\frac{6\pi}{17} + i \sin -\frac{6\pi}{17}\right)}$$

$$\left(\cos -\frac{2\pi}{17} + i \sin -\frac{2\pi}{17}\right) \rightarrow \cos \frac{51\pi}{17} + i \sin \frac{51\pi}{17}$$

$= (\cos 3\pi + i \sin 3\pi)$   
 $= -1$



Express  $(1 + \sqrt{3}i)^7$  in the form  $x + iy$  where  $x, y \in \mathbb{R}$ .

$$\begin{aligned} \sqrt{1^2 + \sqrt{3}^2} &= 2 & (2(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi))^7 \\ \tan^{-1}(\sqrt{3}) &= \frac{1}{3}\pi & 2^7 (\cos \frac{7}{3}\pi + i \sin \frac{7}{3}\pi) \\ && 128 (\cos \frac{7}{3}\pi + i \sin \frac{7}{3}\pi) \\ && 64 + 64\sqrt{3}i \end{aligned}$$

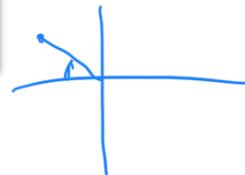
$$z = -8 + (8\sqrt{3})i$$

- (a) Find the modulus of  $z$  and the argument of  $z$ . (3)

Using de Moivre's theorem,

- (b) find  $z^3$ , (2)

0a	Modulus = 16 Argument = $\arctan(-\sqrt{3}) + \frac{2\pi}{3}$	BI MIAI (3)
0b	$z^3 = 16^3(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^3 = 16^3(\cos 2\pi + i \sin 2\pi) = 4096$	MIAI (2)



$$a) \sqrt{8^2 + (8\sqrt{3})^2} = 16$$

$$b) \pi - \tan^{-1}\left(\frac{8\sqrt{3}}{8}\right) = \frac{2\pi}{3}$$

$$b) z = 16\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

$$\begin{aligned} z^3 &= 16^3 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3 \\ &= 4096 \left(\cos 2\pi + i \sin 2\pi\right) \\ &= 4096 \end{aligned}$$

$$\begin{aligned} \cos 2\pi &= 1 \\ \sin 2\pi &= 0 \end{aligned}$$

Ex 1C

## Applications of de Moivre #1: Trig identities

Express  $\cos 3\theta$  in terms of powers of  $\cos \theta$

- 1) Create a 'de Moivre' statement that includes a  $\cos 3\theta$  on RHS
- 2) Binomial expansion
- 3) Compare real/imaginary parts

$$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (\cos \theta + i \sin \theta)^3 \\ &= \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 \\ &= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta \end{aligned}$$

Only take real parts for cos. ie not  $i$  terms

$$\begin{aligned} \cos 3\theta &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta \quad \leftarrow \begin{array}{l} \sin^2 \theta + \cos^2 \theta = 1 \\ \sin^2 \theta = 1 - \cos^2 \theta \end{array} \\ &= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta) \\ &= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta \\ &= 4 \cos^3 \theta - 3 \cos \theta \end{aligned}$$

Express

(a)  $\cos 6\theta$  in terms of  $\cos \theta$ .

(b)  $\frac{\sin 6\theta}{\sin \theta}$ ,  $\theta \neq n\pi$ , in terms of  $\cos \theta$ .

$$\cos 6\theta + i \sin 6\theta = (\cos \theta + i \sin \theta)^6$$

$$= \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - 20i \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta + 6i \cos \theta \sin^5 \theta - \sin^6 \theta$$

Just real parts

$$\begin{aligned}\cos 6\theta &= \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta \\&= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3 \\&= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta (1 - 2\cos^2 \theta + \cos^4 \theta) - (1 - 3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta) \\&= 17 \cos^6 \theta - 18 \cos^4 \theta + 15 \cos^2 \theta - 30 \cos^4 \theta + 15 \cos^6 \theta - 1 + 3 \cos^2 \theta \\&= 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1\end{aligned}$$

b) Imaginary parts divided by  $\sin \theta$

$$\frac{\sin 6\theta}{\sin \theta} = \frac{6 \cos^5 \theta \sin \theta}{\sin \theta} - \frac{20 \cos^3 \theta \sin^3 \theta}{\sin \theta} + \frac{6 \cos \theta \sin^5 \theta}{\sin \theta}$$

$$\begin{aligned}&= 6 \cos^5 \theta - 20 \cos^3 \theta \sin^2 \theta + 6 \cos \theta \sin^4 \theta \\&= 6 \cos^5 \theta - 20 \cos^3 \theta (1 - \cos^2 \theta) + 6 \cos \theta (1 - \cos^2 \theta)^2 \\&= 6 \cos^5 \theta - 20 \cos^3 \theta + 20 \cos^5 \theta + 6 \cos \theta - 12 \cos^3 \theta + 6 \cos^5 \theta \\&= 32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta\end{aligned}$$

(a) Use de Moivre's theorem to show that

$$\sin 5\theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

(5)

Hence, given also that  $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$ 

(b) Find all the solutions of

$$\sin 5\theta = 5 \sin 3\theta$$

in the interval  $0 \leq \theta < 2\pi$ . Give your answers to 3 decimal places. (6)

$$\cos 5\theta + i \sin 5\theta = (\cos \theta + i \sin \theta)^5$$

$$= \cos^5 \theta + 5i \cos^4 \theta \sin \theta - 10 \cos^3 \theta \sin^2 \theta - 10i \cos^2 \theta \sin^3 \theta + 5 \cos \theta \sin^4 \theta + i \sin^5 \theta$$

Just take imaginary

$$\sin 5\theta = 5 \cos^4 \theta \sin \theta - 10 \cos^2 \theta \sin^3 \theta + \sin^5 \theta$$

$$\sin 5\theta = 5(1 - \sin^2 \theta)^2 \sin \theta - 10 \sin^3 \theta (1 - \sin^2 \theta) + \sin^5 \theta$$

$$\sin 5\theta = 5 \sin \theta - 10 \sin^3 \theta + 5 \sin^5 \theta - 10 \sin^3 \theta + 10 \sin^5 \theta + 5 \sin^5 \theta$$

$$= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

$$\text{b) } 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta = 15 \sin \theta - 20 \sin^3 \theta$$

$$16 \sin^5 \theta - 10 \sin \theta = 0$$

$$\sin \theta (8 \sin^4 \theta - 5) = 0$$

$$\sin \theta = 0 \quad \text{or} \quad \sin^4 \theta = 5/8$$

$$\theta = 0, \pi \quad \theta = 2.046, 4.237, 5.188$$

50	$\int \frac{dx}{x^2 + 1} = \arctan x + C$	ME
	$\int \frac{dx}{x^2 + 1} = \frac{1}{2} \int \frac{d(x^2 + 1)}{x^2 + 1} = \frac{1}{2} \ln(x^2 + 1) + C$	ME
	$\int \frac{dx}{x^2 + 1} = \frac{1}{2} \ln(x^2 + 1) + C$	ME
	$\int \frac{dx}{x^2 + 1} = \frac{1}{2} \ln(1 + x^2) + C$	ME
	$\int \frac{dx}{x^2 + 1} = \frac{1}{2} \ln(1 + x^2) + C$	ME

## Finding identities for $\sin^n \theta$ and $\cos^n \theta$

The technique we've seen allows us to write say  $\cos 3\theta$  in terms of powers of  $\cos \theta$  (e.g.  $\cos^3 \theta$ ).

Is it possible to do the opposite, to say express  $\cos^3 \theta$  in terms of a linear combination of  $\cos 3\theta$  and  $\cos \theta$  (with no powers)?

If  $z = \cos \theta + i \sin \theta$ , what is  $z + \frac{1}{z}$  and  $z - \frac{1}{z}$ ?

$$\frac{1}{z} = z^{-1} = \cos -\theta + i \sin -\theta = \cos \theta - i \sin \theta$$

$$z + \frac{1}{z} = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta = 2 \cos \theta$$

$$z - \frac{1}{z} = \cos \theta + i \sin \theta - (\cos \theta - i \sin \theta) = 2i \sin \theta$$

And what is  $z^n + \frac{1}{z^n}$ ?  $z^n - \frac{1}{z^n}$ ?

$$z^n + \frac{1}{z^n} = 2 \cos(n\theta)$$

$$z^n - \frac{1}{z^n} = 2i \sin(n\theta)$$

Express  $\cos^5 \theta$  in the form  $a \cos 5\theta + b \cos 3\theta + c \cos \theta$

- 1) Raise RHS to the required power – careful of the ‘2’ or ‘2i’
- 2) Raise LHS to same power
- 3) Binomial expansion
- 4) Use the identities once again
- 5) Remember to isolate by dividing by any coefficients on LHS

Results you need to use

$$z + \frac{1}{z} = 2 \cos \theta \quad z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z - \frac{1}{z} = 2i \sin \theta \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$

$$(2 \cos \theta)^5 = \left(z + \frac{1}{z}\right)^5$$

$$32 \cos^5 \theta = z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$$

$$32 \cos^5 \theta = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$$

$$32 \cos^5 \theta = 2 \cos 5\theta + 5(2 \cos 3\theta) + 10(2 \cos \theta)$$

$$\cos 5\theta = \frac{1}{16} \cos 5\theta + \frac{5}{16} \cos 3\theta + \frac{5}{8} \cos \theta$$

Prove that  $\sin^3 \theta = -\frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta$

- 1) Raise RHS to the required power – careful of the ‘2’ or ‘2i’
- 2) Raise LHS to same power
- 3) Binomial expansion
- 4) Use the identities once again
- 5) Remember to isolate by dividing by any coefficients on LHS

☞ Results you need to use

$$z + \frac{1}{z} = 2 \cos \theta \quad z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z - \frac{1}{z} = 2i \sin \theta \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$

$$(2i \sin \theta)^3 = \left(z - \frac{1}{z}\right)^3$$

$$-8i \sin^3 \theta = z^3 - 3z + \frac{3}{z} - \frac{1}{z^3}$$

$$-8i \sin^3 \theta = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z + \frac{1}{z}\right)$$

$$-8i \sin^3 \theta = 2i \sin 3\theta - 6i \sin \theta$$

$$\sin^3 \theta = -\frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta$$

## Your Turn

(a) Express  $\sin^4 \theta$  in the form  $a \cos 4\theta + b \cos 2\theta + c$

(b) Hence find the exact value of  $\int_0^{\frac{\pi}{2}} \sin^4 \theta \ d\theta$

$$(2i \sin \theta)^4 = \left(2 - \frac{1}{z}\right)^4$$

$$16 \sin^4 \theta = z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}$$

$$16 \sin^4 \theta = \left(z^4 + \frac{1}{z^4}\right) - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$16 \sin^4 \theta = 2 \cos 4\theta - 8 \cos 2\theta + 6$$

$$\sin^4 \theta = \frac{1}{8} \cos 4\theta - \frac{1}{2} \cos 2\theta + \frac{3}{8}$$

b)  $\int_0^{\frac{\pi}{2}} \frac{1}{8} \cos 4\theta - \frac{1}{2} \cos 2\theta + \frac{3}{8} \ d\theta$

$$\left[ \frac{1}{32} \sin 4\theta - \frac{1}{4} \sin 2\theta + \frac{3}{8}\theta \right]_0^{\frac{\pi}{2}} = \frac{3\pi}{16}$$

Ex 1D

# Sums of Series

The formula for the sum of a geometric series also applies to complex numbers:

For  $w, z \in \mathbb{C}$ ,

$$\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$$

$$\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \cdots + wz^{n-1} = \frac{w}{1 - z}$$

*provided  $|z| < 1$*

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{\infty} = \frac{a}{1 - r}$$

**Geometric series**

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{\infty} = \frac{a}{1 - r} \text{ for } |r| < 1$$

Show that if  $z = e^{\frac{\pi i}{4}}$ , then  $\sum_{r=0}^8 z^r = 1$

Show that  $\sum_{r=0}^5 (1 + i\sqrt{3})^r = -21\sqrt{3}i$

Ex1E Q2, 3

## Some very useful further identities

 Results you need to use

$$z + \frac{1}{z} = 2 \cos \theta \quad z^n + \frac{1}{z^n} = 2 \cos n\theta$$

$$z - \frac{1}{z} = 2i \sin \theta \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$

- 1) Rewrite these in exponential form
- 2) Make  $\sin n\theta$  and  $\cos n\theta$  the subject

Things to note:

- Indices are same but negated
- $\cos n\theta$  goes with +
- $\sin n\theta$  goes with -
- Hyperbolic connection...

# Creating expressions in the hyperbolic form PART 1

$$\cos n\theta = \frac{1}{2}(e^{ni\theta} + e^{-ni\theta})$$

$$\sin n\theta = \frac{1}{2i}(e^{ni\theta} - e^{-ni\theta})$$

$$2\cos n\theta = e^{ni\theta} + e^{-ni\theta}$$

$$2i\sin n\theta = e^{ni\theta} - e^{-ni\theta}$$

when there is a 1+, 1-, or -1,  
with coefficient of  $e^{in\theta}$  as 1

$$\frac{3}{e^{2i\theta} - 1}$$

$$\frac{e^{i\theta}}{e^{\frac{i\theta}{3}} - 1}$$

$$\frac{e^{5i\theta} + 1}{e^{4i\theta} - 1}$$

# Tricky example using several skills we have learned

Given that  $z = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$ , where  $n$  is a positive integer, show that

$$1 + z + z^2 + \cdots + z^{n-1} = 1 + i \cot\left(\frac{\pi}{2n}\right)$$

## Geometric series

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_\infty = \frac{a}{1 - r} \text{ for } |r| < 1$$

Ex1E Q1

# Using mod-arg form to split summation

$e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots + e^{ni\theta}$  is a geometric series,

$$\therefore e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots + e^{ni\theta} = \frac{e^{i\theta}(e^{ni\theta} - 1)}{e^{i\theta} - 1}$$

Converting each exponential term to modulus-argument form would allow us to consider the real and imaginary parts of the series separately:

$$\begin{aligned} e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots + e^{ni\theta} \\ = (\cos \theta + i \sin \theta) + (\cos 2\theta + i \sin 2\theta) + \dots \\ = (\cos \theta + \cos 2\theta + \dots) + i(\sin \theta + \sin 2\theta + \dots) \end{aligned}$$

Thus  $\cos \theta + \cos 2\theta + \dots$  is the real part of  $\frac{e^{i\theta}(e^{ni\theta} - 1)}{e^{i\theta} - 1}$  and  $\sin \theta + \sin 2\theta + \dots$  the imaginary part.

$S = e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \cdots + e^{8i\theta}$ , for  $\theta \neq 2n\pi$ , where  $n$  is an integer.

(a) Show that  $S = \frac{e^{\frac{9i\theta}{2}} - 1}{e^{\frac{i\theta}{2}} - 1} \sin 4\theta$

Let  $P = \cos \theta + \cos 2\theta + \cos 3\theta + \cdots + \cos 8\theta$  and  $Q = \sin \theta + \sin 2\theta + \cdots + \sin 8\theta$

(b) Use your answer to part a to show that  $P = \cos \frac{9\theta}{2} \sin 4\theta \cosec \frac{\theta}{2}$  and find similar expressions for  $Q$  and  $\frac{Q}{P}$

Ex1E Q5  
Rev 1 Q6  
Ex1E Q6 \*hard

## Creating expressions in the hyperbolic form PART 2

$$\cos n\theta = \frac{1}{2}(e^{ni\theta} + e^{-ni\theta})$$

$$\sin n\theta = \frac{1}{2i}(e^{ni\theta} - e^{-ni\theta})$$

$$2\cos n\theta = e^{ni\theta} + e^{-ni\theta}$$

$$2i\sin n\theta = e^{ni\theta} - e^{-ni\theta}$$

when there is a k+, k-, or -k  
where k is a constant, or if  
there is a  $ke^{in\theta}$  instead of  $e^{in\theta}$

multiply by same expression by with power negated

$$\frac{3}{e^{4i\theta} + 2}$$

$$\frac{1}{3e^{i\theta} - 1}$$

$$\frac{e^{i\theta}}{3 - e^{2i\theta}}$$

4. The infinite series C and S are defined by

$$C = \cos \theta + \frac{1}{2} \cos 5\theta + \frac{1}{4} \cos 9\theta + \frac{1}{8} \cos 13\theta + \dots$$

$$S = \sin \theta + \frac{1}{2} \sin 5\theta + \frac{1}{4} \sin 9\theta + \frac{1}{8} \sin 13\theta + \dots$$



Given that the series C and S are both convergent,

(a) show that

$$C + iS = \frac{2e^{i\theta}}{2 - e^{4i\theta}} \quad (4)$$

(b) Hence show that

$$S = \frac{4\sin \theta + 2\sin 3\theta}{5 - 4\cos 4\theta} \quad (4)$$

Ex1E Q4, 7  
Mix 1 Q13

## Applications of de Moivre #2: Roots

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

We have so far used de Moivre's theorem when  $n$  was an integer.  
It also works however when  $n$  is a rational number! (proof not required)

Solve  $z^3 = 1$

Plot these roots on an Argand diagram

Solve  $z^4 = 2 + 2\sqrt{3} i$

### Edexcel FP2(Old) June 2012 Q3

- a) Express the complex number  $-2 + (2\sqrt{3})i$  in the form  $r(\cos \theta + i \sin \theta)$ ,  
 $-\pi < \theta \leq \pi$ . (3)
- b) Solve the equation

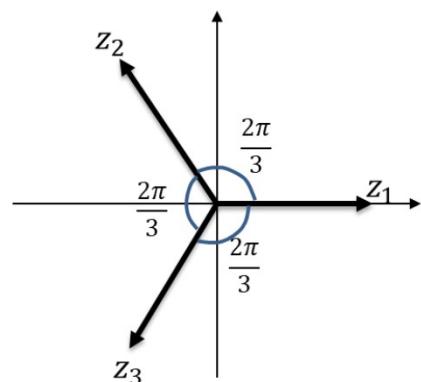
$$z^4 = -2 + (2\sqrt{3})i$$

giving the roots in the form  $r(\cos \theta + i \sin \theta)$ ,  $-\pi < \theta \leq \pi$ . (5)

## Roots of Unity

Solve  $z^3 = 1$

Plot these roots on an Argand diagram



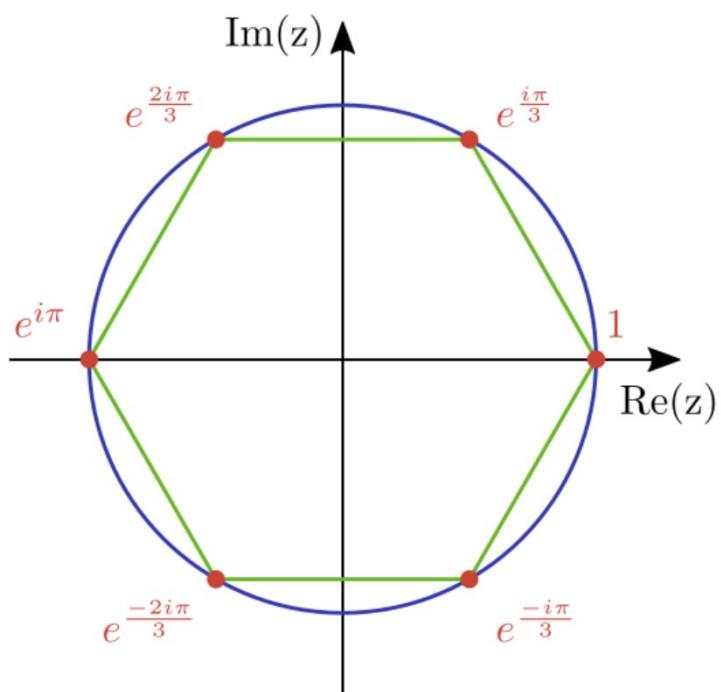
Solve  $z^4 = 1$

Plot these roots on an Argand diagram

Solve  $z^6 = 1$

Plot these roots on an Argand diagram

How would the solutions to  $z^n = 1$  look on an Argand diagram?



- 1 will always be a root
- Suppose the next root around the polygon is  $\omega$ ... then...

The roots of unity sum to zero

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

Solve  $z^5 = 1$

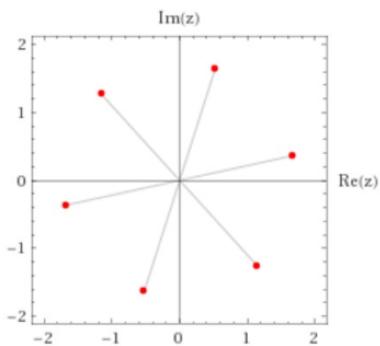
Hence show that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}$$

Ex1F

# Solving Geometric Problems

$$z^6 = 7 + 24i$$



Recall that  $\omega$  is the first root of unity:  $\cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$ .  
This has modulus 1 and argument  $\frac{2\pi}{n}$ .

If  $z_1$  is one root of the equation  $z^n = s$ , and  $1, \omega, \omega^2, \dots, \omega^{n-1}$  are the  $n$ th roots of unity, then the roots of  $z^n = s$  are given by  $z_1, z_1\omega, z_1\omega^2, \dots, z_1\omega^{n-1}$ .

The point  $P(\sqrt{3}, 1)$  lies at one vertex of an equilateral triangle. The centre of the triangle is at the origin.

- (a) Find the coordinates of the other vertices of the triangle.
- (b) Find the area of the triangle.

Find the coordinates of the vertices of an equilateral triangle with centre  $(5, 5)$  and one vertex at  $(3, 4)$

6. In an Argand diagram, the points  $A$ ,  $B$  and  $C$  are the vertices of an equilateral triangle with its centre at the origin. The point  $A$  represents the complex number  $6 + 2i$ .

(a) Find the complex numbers represented by the points  $B$  and  $C$ , giving your answers in the form  $x + iy$ , where  $x$  and  $y$  are real and exact.

The points  $D$ ,  $E$  and  $F$  are the midpoints of the sides of triangle  $ABC$ .

- (b) Find the exact area of triangle  $DEF$ .

$\frac{d}{dx} \ln(x)$	$\frac{d}{dx} \ln(x^2)$	$\frac{d}{dx} \ln(\sin x)$
$\frac{1}{x}$	$\frac{2}{x}$	$\frac{\cos x}{\sin x}$
$\frac{1}{x}$	$\frac{2}{x}$	$\frac{\cos x}{\sin x}$
$\frac{1}{x}$	$\frac{2}{x}$	$\frac{\cos x}{\sin x}$
$\frac{1}{x}$	$\frac{2}{x}$	$\frac{\cos x}{\sin x}$

(3)

Ex1G