Complex Numbers II – Test A (70 mins)

Fundamentals Challenge ✓Expert

Subtopics: Exponential form of complex numbers, multiplying and dividing complex numbers, de Moivre's theorem, trigonometric identities, sums of series, n^{th} roots of a complex number, solving geometric problems

1. a) Use **de Moivre's theorem** to express $x=\left(6\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)^3$ and $y=\left(4\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\right)^4$ in the form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and θ is in terms of π **[2]**

   b) Hence find $\frac{x}{y}$ in the form $a+b\mathrm{i}$, where $a,\ b\in\mathbb{R}$ **[4]**

2. a) Use **de Moivre's theorem** to show that $\cos 5\theta\equiv 16\cos^5\theta-20\cos^3\theta+5\cos\theta$ **[5]**

   b) Find an expression for $\sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$, and hence show that $\tan 5\theta\equiv\frac{5\tan\theta-10\tan^3\theta+\tan^5\theta}{16-20\sec^2\theta+5\sec^4\theta}$ **[4]**

3. a) Given that $z=\cos\theta+\mathrm{i}\sin\theta$, use the results $z^n-\frac{1}{z^n}=2\mathrm{i}\sin n\theta$ and $z^n+\frac{1}{z^n}=2\cos n\theta$ to prove that $\sin^2\theta\cos^2\theta\equiv\frac{1}{8}(1-\cos 4\theta)$ **[5]**

   b) Hence find the **exact** value of $\int_0^{\frac{\pi}{16}}\sin^2\theta\cos^2\theta\ \mathrm{d}\theta$ **[3]**

4. a) Show how $2\sqrt{3}+2\mathrm{i}$ can be expressed in the form $r(\cos\theta+\mathrm{i}\sin\theta)$, where $r>0$ and $-\pi<\theta\le\pi$. Give your values of r and θ **exactly**. **[3]**

   b) Hence solve the equation $z^4=2\sqrt{3}+2\mathrm{i}$, giving your answers in the form $z=r\mathrm{e}^{\mathrm{i}n\pi}$, where each n is a rational number such that $|n|\le 1$ **[7]**

   c) Illustrate the roots of this equation on an Argand diagram **[1]**

5. a) Given that the **infinite series** A and B defined by $A=1-\frac{1}{3}\cos 2\theta+\frac{1}{9}\cos 4\theta-\frac{1}{27}\cos 6\theta+\ldots$ and $B=\frac{1}{3}\sin 2\theta-\frac{1}{9}\sin 4\theta+\frac{1}{27}\sin 6\theta+\ldots$ are **convergent**, show that $A-\mathrm{i}B=\frac{3}{3+\mathrm{e}^{2\mathrm{i}\theta}}$ **[4]**

   b) Hence show that $A=\frac{9+3\cos 2\theta}{10+6\cos 2\theta}$, and find a corresponding expression for B **[5]**

6. Given that $z=6-2\mathrm{i}\sqrt{3}$, $\left|\frac{z^2}{w}\right|=2|z|$ and $\mathrm{Im}\left(\frac{z^2}{w}\right)=0$ where $w\ne 0$, find the **two** possibilities for w in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ is in simplified surd form and $-\pi<\theta\le\pi$ **[6]**

7. Find the **smallest** positive integer n such that $\left(\frac{24}{1-\mathrm{i}\sqrt{3}}\right)^n$ is **real and positive.** **[6]**

8. An **equilateral triangle** has centre (4, 1) and one vertex at (6, 3). Find the coordinates of the other two vertices. **[7]**

TOTAL 62 MARKS

Subtopics: The method of differences, higher derivatives, Maclaurin series, series expansions of compound functions

1. a) Show that $r(3r+1) \equiv r(r+1)^2 - (r-1)r^2$ **[3]**

   b) Hence show that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ using the **method of differences** **[3]**

2. a) Find the first **three non-zero** terms of sin $2x$ and the first **four non-zero** terms of cos $2x$ in their Maclaurin series expansions **[4]**

   b) Hence show that $x\sin 2x - 3\cos 2x = -3 + 8x^2 - \frac{10}{3}x^4 + \frac{8}{15}x^6 + \ldots$ **[2]**

   c) Hence find $\lim_{x \to 0}\left(\frac{x\sin 2x - 3\cos 2x + 3}{x^2}\right)$ **[1]**

3. a) Express $\frac{3}{(3r+1)(3r+4)}$ in the form $\frac{A}{3r+1} + \frac{B}{3r+4}$, where A and B are constants to be found **[2]**

   b) Hence find the **exact** value of $\sum_{r=1}^{30} \frac{3}{(3r+1)(3r+4)}$ **[2]**

4. a) Use the Maclaurin series of e^x to find the first **three non-zero** terms in the expansion of $\frac{1}{2}e^{-2x^2}$ **[3]**

   b) State the range of values of x for which this series expansion is valid **[1]**

   c) Use your answer from part a) and a suitable choice of x to find an approximation for $e^{-0.02}$ **[3]**

5. a) Find the series expansion of $\ln(1-2x)$ **up to and including** the term in x^5, and state the range of values of x for which this expansion is valid. **[3]**

   b) Use your answer to part a) to find an **approximation** for $\ln\left(\frac{2}{3}\right)$, giving your answer to **3 decimal places.** **[3]**

6. Using **differentiation**, find the series expansion of $\ln(1+e^{2x})$ **up to and including** the term in x^2 **[4]**

7. a) Express $x^2 + 8x + 16$ in the form $\left(k\left(1+\frac{x}{4}\right)\right)^2$, where k is a positive constant to be found. **[2]**

   b) Using the Maclaurin series for $\ln(1+x)$, find the series expansion of $\ln(x^2+8x+16)$ **up to and including** the term in x^3, and state the range of values of x for which this series expansion is valid. **[5]**

8. Show that $\frac{1}{1\times 4} + \frac{1}{2\times 5} + \frac{1}{3\times 6} + \ldots + \frac{1}{n(n+3)} = \frac{11}{18} - \frac{an^2+bn+c}{3(n+1)(n+2)(n+3)}$, where a, b and c are constants to be found. **[6]**

9. a) Show that $e^{2x}\cos ax = 1 + 2x + \frac{x^2}{2}(4-a^2) + \frac{x^3}{3}(4-3a^2) + \ldots$, where a is a constant **[4]**

   b) Given that a is **positive** and that the **first non-zero** term in the series expansion (in ascending powers of x) of $e^{2x}\cos ax + \frac{6\sin bx}{x} - 2x - 4$ is cx^3, find the **exact** values of a, b and c **[5]**

TOTAL 56 MARKS

Further Calculus – Test A (45 mins)

Fundamentals Challenge ✓ **Expert**

Subtopics: Improper integrals, the mean value of a function, differentiating inverse trigonometric functions, integrating with inverse trigonometric functions, integrating using partial fractions, volumes of revolution around the x-axis and the y-axis, volumes of revolution of parametrically defined curves, modelling with volumes of revolution

1. a) Find $\int \frac{1}{25+x^2}\,dx$ **[1]**

   b) Find $\int \frac{1}{25+9x^2}\,dx$ in the form $A\arctan(Bx)+c$, where A and B are constants to be found, and c is an arbitrary constant **[4]**

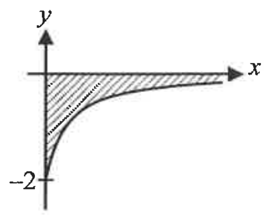
2. a) Use **a suitable substitution** to find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$ in the form $A\arcsin(Bx)+c$, where A and B are constants to be found and c is an arbitrary constant **[5]**

   b) Find the **exact mean value** of $f(x)=\frac{1}{\sqrt{1-9x^2}}$ over the interval $\left[0,\ \frac{1}{6}\right]$ **[3]**

   c) Hence state the **exact mean value** of $-f(x)$ over the interval $\left[0,\ \frac{1}{6}\right]$ **[1]**

3. a) State why the integral $\int_0^{\infty} e^{-x}\,dx$ is **improper**. **[1]**

   b) The diagram to the right shows the curve with equation $y=-2e^{-x},\ x\geq 0$. Find the **area** of the shaded region bounded by the curve and the coordinate axes. **[4]**



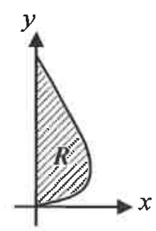
4. $f(x)=\frac{2x^2-3x+16}{(x-2)(x^2+5)}$

   a) Express $f(x)$ in the form $\frac{A}{x-2}+\frac{B}{x^2+5}$, where A and B are constants to be found. **[3]**

   b) Hence show that $\int_2^6 f(x)\,dx$ **diverges**. **[5]**

5. Find the equation of the **tangent** to the curve $y=\arccos\left(\frac{x}{2}\right)$ at the point where $x=1$. Give your answer in the form $ay=b\pi+\sqrt{3}(1-x)$, where a and b are constants to be found. **[6]**

6. The diagram to the right shows the curve C with parametric equations $x=\frac{1}{10}\sin t,\ y=\frac{1}{100}t^3,\ 0\leq t\leq \pi$. A large candle is modelled as the solid of revolution formed when the shaded region R is rotated 2π radians about the y-axis. Wax is melted down and poured into a mould to form each candle.
   Given that the units of x and y are metres, how many whole candles can be made from 1 m^3 of wax? **[8]**

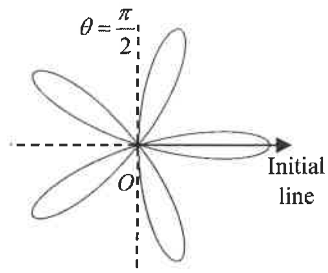


TOTAL 41 MARKS

Subtopics: Polar coordinates and equations, sketching curves, area enclosed by a polar curve, tangents to polar curves

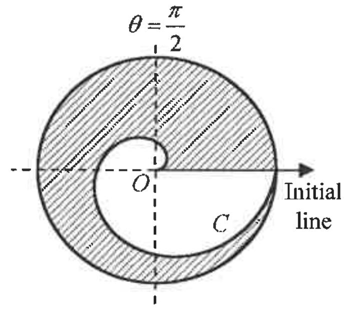
1. On separate diagrams, sketch the following curves for $0 \le \theta < 2\pi$:
   a) $r = 1 + \sin\theta$ **[2]**
   b) $r = 2 - \cos\theta$ **[2]**
   c) $r^2 = 4\sin 2\theta$ **[2]**

2. The curve with polar equation $r = 2\cos 5\theta$ is sketched to the right.
   Find the **exact** area of one of the loops of this curve. **[4]**

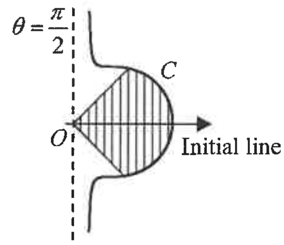


3. The curve C has equation $r = 4 - 4\cos\theta$ for $0 < \theta < \pi$
   a) Find the exact **Cartesian** coordinates of the point on C where the tangent is **parallel** to the initial line. **[6]**
   b) Find a **polar equation** for the tangent to C that is **perpendicular** to the initial line. Give your answer in the form $r = f(\theta)$. **[6]**

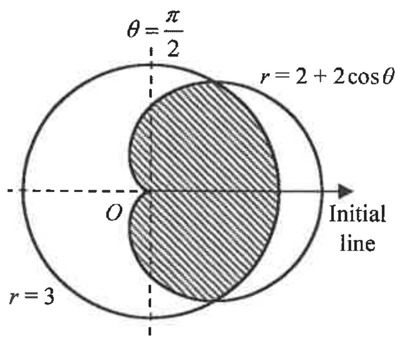
4. The curve C has equation $r = k\theta$ for some constant $k > 0$.
   The diagram to the right shows the curve C and the circle with radius 1, where both curves are defined for $0 < \theta \le 2\pi$.
   The two curves meet on the initial line.
   a) Find the exact value of k **[1]**
   b) Hence find the exact area of the shaded region bounded by the circle and the curve C between $\theta = 0$ and $\theta = 2\pi$ **[5]**



5. The curve C with equation $r = a\cos\theta + \sec\theta$ is sketched to the right, where a is a rational number and $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$.
   The shaded region shown to the right is bounded by C and the half-lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$. Given that this shaded region has area $17 + 12\pi$, find the value of a. **[6]**



6. The sketch to the right shows the curves with polar equations $r = 2 + 2\cos\theta$ and $r = 3$ for $0 \le \theta < 2\pi$
   a) Find the **exact** polar coordinates of the two points where these curves intersect. **[3]**
   b) Find the area of the shaded region that is contained within both curves. Give your answer in the form $a\pi + b\sqrt{3}$, where a and b are **rational** numbers. **[7]**



TOTAL 44 MARKS

Hyperbolic Functions – Test A (65 mins)

Subtopics: Hyperbolic functions, inverse hyperbolic functions, identities and equations, differentiating hyperbolic functions, integrating hyperbolic functions

1. Differentiate each of the following with respect to x:

   a) $\cosh 2x$ b) $\frac{1}{4}\tanh 4x$ c) $x\sinh 3x$ d) $\text{arcosh}\, 3x$ **[6]**

2. Find:

   a) $\int \frac{3}{\sqrt{x^2+1}}\,\mathrm{d}x$ b) $\int \frac{4}{\sqrt{x^2-25}}\,\mathrm{d}x$ **[2]**

3. On the **same diagram**, sketch the graphs of $y = \sinh x$ and $y = 3 + \sinh 4x$ **[3]**

4. Solve the equation $5\cosh x + 3\sinh x = 5$, giving your answers **exactly**. **[6]**

5. Sketch the graph of $y = \text{sech}\, x$ **[2]**

6. a) By writing $\text{artanh}\, x = u$, show that $u = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$ **[4]**

   b) Given that $\text{artanh}\, x - \text{artanh}\, y = \ln\sqrt{5}$, show that $y = \frac{3x-2}{3-2x}$ **[6]**

7. Use the substitution $x = \sqrt{\sinh u}$ to show that $\int \frac{2x}{(1+x^4)^{3/2}}\,\mathrm{d}x = \frac{x^2}{\sqrt{1+x^4}} + c$ **[6]**

8. Use the substitution $x = \frac{1}{8}(3 + 5\cosh u)$ to find $\int \frac{1}{\sqrt{4x^2 - 3x - 1}}\,\mathrm{d}x$ **[6]**

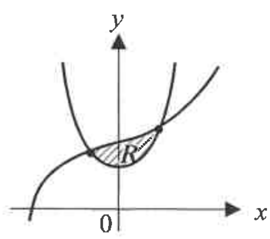
9. Given that $9\cosh x + 12\sinh x = R\sinh(x+\alpha)$, $R > 0$, find the **exact** value of R and the value of α to **3 significant figures**. **[5]**

10. Find the equation of the tangent to $y = \text{arsinh}\left(\frac{x}{4}\right)$ at the point where $x = 3$, giving your answer in the form $y = px + q + \ln r$, where p, q and r are **real numbers**. **[5]**

11. The sketch to the right shows the region R bounded by the curves $y = 7 + \sinh x$ and $y = 5\cosh x$

    a) Find the **exact x-coordinates** of the two points where the curves intersect. **[6]**

    b) Find the area of the shaded region R, giving your answer in the form $y = a\ln b + c$, where a, b and c are integers. **[6]**



TOTAL 63 MARKS

Differential Equations – Test A (80 mins)

Fundamentals Challenge ✓**Expert**

Subtopics: First- and second-order differential equations, boundary conditions, modelling, simple harmonic motion, damped and forced harmonic motion, coupled first-order simultaneous differential equations

1. a) Find the **general solution** to the differential equation $y'' - 4y' - 5y = 0$, where y is a function of x **[3]**

   b) Hence find the **general solution** to the differential equation $y'' - 4y' - 5y = 18e^{-x}$ **[5]**

2. Find the **general solution** to the differential equation $\dfrac{dy}{dx} - y\tan x = x$ in the form $y = f(x)$ **[8]**

3. A particle moves in a straight line with simple harmonic motion. At time t seconds the particle's displacement in metres from the point O is described by the differential equation $\ddot{x} = -4x$.
   When $t = 0$, $x = 3$ and $\dot{x} = -8$.

   a) Find an expression for x in terms of t **[7]**

   b) Hence find the **maximum distance** of the particle from the point O and the **period** of its motion **[4]**

4. A rock is eroded at an increasing rate over time and is also being consumed by a colony of bacteria. After t centuries, the height of the rock is x cm and the number of bacteria in the colony, measured in millions, is y. The rates of change of x and y are given, respectively, by the equations:

$$\frac{dx}{dt} = -y - 4t$$

$$\frac{dy}{dt} = 4x + 4y$$

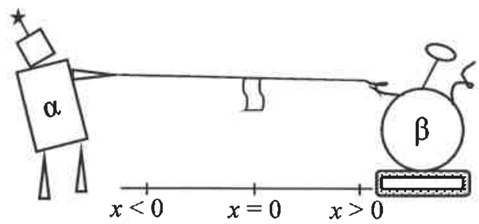
   a) Find a **general solution** for x in terms of t **[11]**

   When $t = 0$, the rock is 123 cm tall and there are 16 million bacteria in the colony.

   b) Find the **particular solution** for y at time t **[4]**

   c) Explain why this model becomes **unsuitable** for large t **[1]**

5. Two robots, Alphatron and Betabot, are competing in a tug of war. The rope starts at rest. After t seconds, the centre of the rope has a displacement of x metres from its starting position towards Betabot, where x is described by the differential equation $\ddot{x} + 2\dot{x} + 5x = 2\cos t$. Whichever robot the centre of the rope is closer to after three seconds wins the tug of war. Who wins the tug of war? **[12]**



6. A meteor of mass 10 kg is falling vertically through the atmosphere. Its height above the ground at time t seconds is x metres. The only forces acting on the meteor are its weight vertically downwards and a drag force, D, due to air resistance vertically upwards. When the meteor is travelling with velocity v m s^{-1} downwards, the drag force has magnitude $D = v$ N. Take $\mathbf{g = 10}$ **m s^{-2}**.

   a) Explain why $v = -\dfrac{dx}{dt}$, and hence show that, while it is falling, the height of the meteor above the ground at time t is modelled by the differential equation $10\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = -100$ **[3]**

   When $t = 0$, the meteor is travelling at 10 m s^{-1} downwards.

   b) Determine the **speed** of the meteor in m s^{-1} after 5 seconds correct to **1 decimal place**. **[9]**

TOTAL 67 MARKS

1. a) $x=\left(6\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)^3=6^3\left(\cos\left(3\times\frac{\pi}{6}\right)+\mathrm{i}\sin\left(3\times\frac{\pi}{6}\right)\right)$

$=216\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$ by de Moivre's theorem **A1**

$y=\left(4\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)\right)^4=4^4\left(\cos\left(4\times\frac{2\pi}{3}\right)+\mathrm{i}\sin\left(4\times\frac{2\pi}{3}\right)\right)$

$=256\left(\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}\right)$ by de Moivre's theorem **A1**

b) $\frac{x}{y}=\frac{216\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)}{256\left(\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}\right)}$

This is of the form $\frac{z_1}{z_2}$ with $z_1=216\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$ and $z_2=256\left(\cos\frac{8\pi}{3}+\mathrm{i}\sin\frac{8\pi}{3}\right)$

So $\frac{z_1}{z_2}=\frac{216\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}}{256\mathrm{e}^{\frac{8\pi}{3}\mathrm{i}}}$ **M1**

Technique: Write both the numerator and the denominator in the form $re^{i\theta}$. Then divide the moduli and subtract the arguments to find the result of the division.

$=\frac{216}{256}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\frac{8\pi}{3}\right)}=\frac{27}{32}\mathrm{e}^{-\frac{13\pi}{6}\mathrm{i}}$ **M1**

$=\frac{27}{32}\left(\cos\left(-\frac{13\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{13\pi}{6}\right)\right)$ **M1**

$=\frac{27\sqrt{3}}{64}-\frac{27}{64}\mathrm{i}$ **A1** **[6 Marks]**

2. a) **Show that $\cos 5\theta\equiv 16\cos^5\theta-20\cos^3\theta+5\cos\theta$**

$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos 5\theta+\mathrm{i}\sin 5\theta$ by de Moivre's theorem **M1**

Technique: Expand the left-hand side using the binomial expansion and simplify the powers of i

$(\cos\theta+\mathrm{i}\sin\theta)^5=\cos^5\theta+{}^5C_1\cos^4\theta\times\mathrm{i}\sin\theta+{}^5C_2\cos^3\theta\times(\mathrm{i}\sin\theta)^2+{}^5C_3\cos^2\theta\times(\mathrm{i}\sin\theta)^3$

$+{}^5C_4\cos\theta\times(\mathrm{i}\sin\theta)^4+(\mathrm{i}\sin\theta)^5$ **M1**

$=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta+10\mathrm{i}^2\cos^3\theta\sin^2\theta+10\mathrm{i}^3\cos^2\theta\sin^3\theta+5\mathrm{i}^4\cos\theta\sin^4\theta+\mathrm{i}^5\sin^5\theta$

$=\cos^5\theta+5\mathrm{i}\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10\mathrm{i}\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+\mathrm{i}\sin^5\theta$ **A1**

Equating real parts:

Tip: Write c for cos θ and s for sin θ while doing this algebra to save space and time

$\cos 5\theta\equiv\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

$\equiv\cos^5\theta-\sin^2\theta\left(10\cos^3\theta-5\cos\theta\sin^2\theta\right)$

$\equiv\cos^5\theta-\left(1-\cos^2\theta\right)\left(10\cos^3\theta-5\cos\theta\left(1-\cos^2\theta\right)\right)$ **M1**

Technique: Use the identity $\cos^2\theta+\sin^2\theta\equiv 1$

$\equiv\cos^5\theta+\left(\cos^2\theta-1\right)\left(10\cos^3\theta-5\cos\theta+5\cos^3\theta\right)$

$\equiv\cos^5\theta+\left(\cos^2\theta-1\right)\left(15\cos^3\theta-5\cos\theta\right)$

$\equiv\cos^5\theta+15\cos^5\theta-5\cos^3\theta-15\cos^3\theta+5\cos\theta$

$\equiv 16\cos^5\theta-20\cos^3\theta+5\cos\theta$ **A1**

b) **Show that** $\tan 5\theta \equiv \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{16 - 20\sec^2\theta + 5\sec^4\theta}$

Equating imaginary parts from a):

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ **A1**

$$\tan 5\theta \equiv \frac{\sin 5\theta}{\cos 5\theta} \equiv \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta} \quad \textbf{M1}$$

$$\equiv \frac{\frac{1}{\cos^5\theta}\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right)}{\frac{1}{\cos^5\theta}\left(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\right)} \quad \textbf{M1}$$

$$\equiv \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{16 - 20\sec^2\theta + 5\sec^4\theta} \quad \textbf{A1}$$ **[9 Marks]**

3. a) **Given that** $z = \cos\theta + i\sin\theta$, **use the results** $z^n - \frac{1}{z^n} = 2i\sin n\theta$ **and** $z^n + \frac{1}{z^n} = 2\cos n\theta$ **to prove that** $\sin^2\theta\cos^2\theta \equiv \frac{1}{8}(1 - \cos 4\theta)$

For $n = 1$, $z - \frac{1}{z} = 2i\sin\theta$ and $z + \frac{1}{z} = 2\cos\theta$

So $\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2 = (2i\sin\theta)^2(2\cos\theta)^2 = -16\sin^2\theta\cos^2\theta$ **M1**

Also $\left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^2 = \left(z^2 - 2z\left(\frac{1}{z}\right) + \left(-\frac{1}{z}\right)^2\right)\left(z^2 + 2z\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2\right)$ **M1**

Technique: Expand the left-hand side using the binomial expansion and simplify the powers of z

$$= \left(z^2 - 2 + \frac{1}{z^2}\right)\left(z^2 + 2 + \frac{1}{z^2}\right)$$

$$= z^4 - 2z^2 + 1 + 2z^2 - 4 + \frac{2}{z^2} + 1 - \frac{2}{z^2} + \frac{1}{z^4}$$

$= z^4 + \frac{1}{z^4} - 2$ **M1**

$= 2\cos 4\theta - 2$ by using $n = 4$ **M1**

$\therefore -16\sin^2\theta\cos^2\theta \equiv 2\cos 4\theta - 2$

$\therefore \sin^2\theta\cos^2\theta \equiv -\frac{1}{16}(2\cos 4\theta - 2) \equiv \frac{1}{8}(1 - \cos 4\theta)$ **A1**

b) $$\int_0^{\frac{\pi}{16}} \sin^2\theta\cos^2\theta \, d\theta = \int_0^{\frac{\pi}{16}} \frac{1}{8}(1 - \cos 4\theta) \, d\theta$$

$= \left[\frac{1}{8}\theta - \frac{1}{32}\sin 4\theta\right]_0^{\frac{\pi}{16}}$ **M1**

$= \frac{1}{8} \times \frac{\pi}{16} - \frac{1}{32}\sin\frac{4\pi}{16} - 0 + \frac{1}{32}\sin 0$ **M1**

$= \frac{\pi}{128} - \frac{\sqrt{2}}{64} = \frac{1}{128}\left(\pi - 2\sqrt{2}\right)$ **A1** **[8 Marks]**

4. a) Let $w = 2\sqrt{3} + 2i$, so $|w| = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$ **M1**

$\arg w = \arctan\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$ **M1**

$\therefore w = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ **A1**

Tip: You should be able to check this result by converting to (r, θ) form using your calculator

b) $z^4 = 2\sqrt{3} + 2\mathrm{i} = 4\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$

$= 4\left(\cos\left(\frac{\pi}{6} + 2k\pi\right) + \mathrm{i}\sin\left(\frac{\pi}{6} + 2k\pi\right)\right)$ **M1**

> **Technique:** Use the facts that $\cos\theta = \cos(\theta + 2k\pi)$ and $\sin\theta = \sin(\theta + 2k\pi)$ for any integer k

Also $z^4 = \left(r(\cos\theta + \mathrm{i}\sin\theta)\right)^4 = r^4(\cos 4\theta + \mathrm{i}\sin 4\theta)$ by de Moivre's theorem **M1**

So $4 = r^4 \therefore r = \sqrt{2}$ and $\frac{\pi}{6} + 2k\pi = 4\theta$ **M1**

> **Technique:** Compare the modulus and argument on the left-hand side and right-hand side for different values of k

For $k = 0$, $\theta = \frac{\pi}{24}$, so $z_1 = \sqrt{2}e^{\mathrm{i}\frac{\pi}{24}}$ **A1**

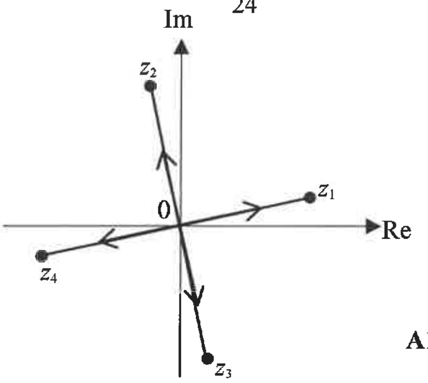
For $k = 1$, $\theta = \frac{13\pi}{24}$, so $z_2 = \sqrt{2}e^{\mathrm{i}\frac{13\pi}{24}}$ **A1**

For $k = -1$, $\theta = -\frac{11\pi}{24}$, so $z_3 = \sqrt{2}e^{-\mathrm{i}\frac{11\pi}{24}}$ **A1**

For $k = -2$, $\theta = -\frac{23\pi}{24}$, so $z_4 = \sqrt{2}e^{-\mathrm{i}\frac{23\pi}{24}}$ **A1**

> **Alternatively:** You can choose $k = 2$, which gives $\theta = \frac{25\pi}{24}$, but you will then have to subtract 2π to produce a value of θ in the interval $-\pi \le \theta \le \pi$

c)



A1 **[11 Marks]**

5. a) **Show that $A - \mathrm{i}B = \frac{3}{3 + e^{2\mathrm{i}\theta}}$**

$A = 1 - \frac{1}{3}\cos 2\theta + \frac{1}{9}\cos 4\theta - \frac{1}{27}\cos 6\theta + \ldots$ and $B = \frac{1}{3}\sin 2\theta - \frac{1}{9}\sin 4\theta + \frac{1}{27}\sin 6\theta + \ldots$

So $A - \mathrm{i}B = \left(1 - \frac{1}{3}\cos 2\theta + \frac{1}{9}\cos 4\theta - \frac{1}{27}\cos 6\theta + \ldots\right) - \mathrm{i}\left(\frac{1}{3}\sin 2\theta - \frac{1}{9}\sin 4\theta + \frac{1}{27}\sin 6\theta + \ldots\right)$

$= 1 - \frac{1}{3}(\cos 2\theta + \mathrm{i}\sin 2\theta) + \frac{1}{3^2}(\cos 4\theta + \mathrm{i}\sin 4\theta) - \frac{1}{3^3}(\cos 6\theta + \mathrm{i}\sin 6\theta) + \ldots$ **M1**

$= 1 - \frac{1}{3}e^{2\mathrm{i}\theta} + \frac{1}{3^2}e^{4\mathrm{i}\theta} - \frac{1}{3^3}e^{6\mathrm{i}\theta} + \ldots$ **M1**

This is an infinite geometric series with $a = 1$ and $r = -\frac{1}{3}e^{2\mathrm{i}\theta}$

So $S_\infty = \frac{1}{1 - \left(-\frac{1}{3}e^{2\mathrm{i}\theta}\right)} = \frac{1}{1 + \frac{1}{3}e^{2\mathrm{i}\theta}}$ **M1**

> **Technique:** Remember that the sum of a convergent infinite geometric series with first term a and common ratio r is $S_\infty = \frac{a}{1-r}$

$= \frac{3}{3 + e^{2\mathrm{i}\theta}}$ **A1**

b) **Show that $A=\dfrac{9+3\cos 2\theta}{10+6\cos 2\theta}$ and find a corresponding expression for B**

$$\frac{3}{3+e^{2i\theta}}=\frac{3}{3+\cos 2\theta+i\sin 2\theta}$$

Technique: Use Euler's relation and conjugates to produce a real denominator

$$=\frac{3}{3+\cos 2\theta+i\sin 2\theta}\times\frac{3+\cos 2\theta-i\sin 2\theta}{3+\cos 2\theta-i\sin 2\theta}=\frac{9+3\cos 2\theta-3i\sin 2\theta}{(3+\cos 2\theta)^2-i^2\sin^2 2\theta}\quad \textbf{M1}$$

Hint: Use the identity $\cos^2 2\theta+\sin^2 2\theta\equiv 1$

$$=\frac{9+3\cos 2\theta-3i\sin 2\theta}{9+6\cos 2\theta+\cos^2 2\theta+\sin^2 2\theta}\quad \textbf{M1}$$

$$=\frac{9+3\cos 2\theta-3i\sin 2\theta}{10+6\cos 2\theta}\quad \textbf{M1}$$

$$\therefore A=\text{Re}\left(\frac{3}{3+e^{2i\theta}}\right)=\frac{9+3\cos 2\theta}{10+6\cos 2\theta}\quad \textbf{A1}$$

Similarly, $B=-\text{Im}\left(\dfrac{3}{3+e^{2i\theta}}\right)=\dfrac{3\sin 2\theta}{10+6\cos 2\theta}$ **A1** **[9 Marks]**

6. $z=6-2i\sqrt{3}$ so $|z|=\sqrt{6^2+\left(-2\sqrt{3}\right)^2}=\sqrt{48}=4\sqrt{3}$ and $\arg z=-\arctan\left(\dfrac{2\sqrt{3}}{6}\right)=-\dfrac{\pi}{6}$

$\therefore z=4\sqrt{3}e^{-\frac{\pi}{6}i}$ **M1**

Let $w=re^{i\theta}$

So $\dfrac{z^2}{w}=\dfrac{\left(4\sqrt{3}\right)^2}{r}e^{\left(2\left(\frac{-\pi}{6}\right)-\theta\right)i}$

$\left|\dfrac{z^2}{w}\right|=\dfrac{\left(4\sqrt{3}\right)^2}{r}=\dfrac{48}{r}$ and $2|z|=2\times 4\sqrt{3}=8\sqrt{3}$ **M1**

Technique: By de Moivre's theorem, the modulus of z^2 is the square of the modulus of z, and the argument of z^2 is twice the argument of z

We are told that these are equal; therefore, $\dfrac{48}{r}=8\sqrt{3}\;\therefore r=2\sqrt{3}$ **M1**

So $\dfrac{z^2}{w}=8\sqrt{3}\,e^{\left(2\left(\frac{-\pi}{6}\right)-\theta\right)i}$

$\text{Im}\left(\dfrac{z^2}{w}\right)=8\sqrt{3}\sin\left(2\times\left(-\dfrac{\pi}{6}\right)-\theta\right)=8\sqrt{3}\sin\left(-\dfrac{\pi}{3}-\theta\right)$ **M1**

Technique: The imaginary part of a complex number is the coefficient of i, i.e. $a\sin b$, where a and b are its modulus and argument respectively

We are told that this is equal to zero, so $-\dfrac{\pi}{3}-\theta=-\pi$ or 0 or π… $\left(\text{i.e. } -\dfrac{\pi}{3}-\theta=n\pi, \text{ where } n \text{ is an integer}\right)$

This gives $\theta=\dfrac{2\pi}{3}$ or $-\dfrac{\pi}{3}$ since $-\pi<\theta\le\pi$ **M1**

$\therefore w=2\sqrt{3}e^{\frac{2\pi}{3}i}$ or $2\sqrt{3}e^{-\frac{\pi}{3}i}$ **A1** **[6 Marks]**

7. $\dfrac{24}{1-i\sqrt{3}}=\dfrac{24}{1-i\sqrt{3}}\times\dfrac{1+i\sqrt{3}}{1+i\sqrt{3}}=\dfrac{24+24i\sqrt{3}}{1-3i^2}$ **M1**

$=\dfrac{24+24i\sqrt{3}}{4}=6+6i\sqrt{3}$ **A1**

$\left|6+6i\sqrt{3}\right|=\sqrt{6+\left(6\sqrt{3}\right)^2}=\sqrt{144}=12$ and $\arg\left(6+6i\sqrt{3}\right)=\arctan\left(\dfrac{6\sqrt{3}}{6}\right)=\dfrac{\pi}{3}$

$\therefore 6+6i\sqrt{3}=12e^{\frac{\pi}{3}i}$ **M1**

So $\left(\dfrac{24}{1-i\sqrt{3}}\right)^n=\left(6+6i\sqrt{3}\right)^n=12^n e^{\frac{n\pi}{3}i}$

$=12^n\left(\cos\dfrac{n\pi}{3}+i\sin\dfrac{n\pi}{3}\right)$ **M1**

This is real and positive when $\sin\frac{n\pi}{3} = 0$ and $\cos\frac{n\pi}{3} > 0$

Alternatively: Consider how many multiples of $\frac{\pi}{3}$ are needed to return to the positive real axis

$\sin\frac{n\pi}{3} = 0$ means $\frac{n\pi}{3} = k\pi$, where k is an integer

For $k = 1$, $n = 3$ and $\sin \pi = 0$, but $\cos \pi = -1$ (so the number is real and negative) **M1**

For $k = 2$, $n = 6$, $\sin 2\pi = 0$ and $\cos 2\pi = 1$ so the number is real and positive $\therefore$ $n = 6$ is the smallest positive integer **A1**

[6 Marks]

8. For this triangle, one vertex is $V_1 = (6, 3)$ and its centre is (4, 1)

So consider $V_1' = (2, 2)$, which represents $2\sqrt{2}e^{\frac{\pi}{4}i}$ **M1**

Technique: Consider the same equilateral triangle with its centre at the origin by translating each point by $\begin{pmatrix}-4\\-1\end{pmatrix}$. Remember to translate the vertices back!

The cube roots of unity are 1, ω and ω^2, where $\omega = e^{\frac{2\pi}{3}i}$

So the other vertices of the translated triangle are given by:

$V_2' = 2\sqrt{2}e^{i\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)} = 2\sqrt{2}e^{\frac{11\pi}{12}i}$, i.e. $\left(-1-\sqrt{3},\ -1+\sqrt{3}\right)$ **M1A1**

Technique: Find the other translated vertices by repeatedly multiplying by $2\sqrt{2}e^{\frac{\pi}{4}i}$ by ω and ω^2

$V_3' = 2\sqrt{2}e^{i\left(\frac{4\pi}{3}+\frac{\pi}{4}\right)} = 2\sqrt{2}e^{\frac{19\pi}{12}i}$, i.e. $\left(-1+\sqrt{3},\ -1-\sqrt{3}\right)$ **M1A1**

So $V_2 = \left(3-\sqrt{3},\ \sqrt{3}\right)$ **A1**

$V_3 = \left(3+\sqrt{3},\ -\sqrt{3}\right)$ **A1**

[7 Marks]

TOTAL 62 MARKS

1. a) **Show that $r(3r+1) \equiv r(r+1)^2 - (r-1)r^2$**

$r(r+1)^2 - (r-1)r^2 = r(r^2+2r+1) - r^3 + r^2$ **M1**

$= r^3 + 2r^2 + r - r^3 + r^2 = 3r^2 + r$ **M1**

$= r(3r+1)$ **A1**

> **Technique:** Expand and simplify the right-hand side

b) **Show that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$, using the method of differences**

$\sum_{r=1}^{n} r(3r+1) = \sum_{r=1}^{n} \left(r(r+1)^2 - (r-1)r^2\right)$

Let $r=1$: $1(2)^2 - 0(1)^2$

$r=2$: $2(3)^2 - 1(2)^2$

$r=3$: $3(4)^2 - 2(3)^2$

$\vdots$

$r=n$: $n(n+1)^2 - (n-1)(n)^2$ **M1M1**

> **Hint:** Write out enough unsimplified terms of the sum to see which ones cancel and which ones remain

So the sum is $n(n+1)^2$

So $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ **A1** **[6 Marks]**

2. a) $\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots$ **M1**

> **Technique:** Replace x with $2x$ in the series expansions for $\sin x$ and $\cos x$

$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \ldots$ **A1**

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots$ **M1**

$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots$ **A1**

b) **Show that $x\sin 2x - 3\cos 2x = -3 + 8x^2 - \frac{10}{3}x^4 + \frac{8}{15}x^6 + \ldots$**

$x\sin 2x - 3\cos 2x = x\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 - \ldots\right) - 3\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \ldots\right)$

$= 2x^2 - \frac{4}{3}x^4 + \frac{4}{15}x^6 - 3 + 6x^2 - 2x^4 + \frac{4}{15}x^6 + \ldots$ **M1**

$= -3 + 8x^2 - \frac{10}{3}x^4 + \frac{8}{15}x^6 + \ldots$ **A1**

c) $\frac{x\sin 2x - 3\cos 2x + 3}{x^2} = 8 - \frac{10}{3}x^2 + \frac{8}{15}x^4 + \ldots$

So $\lim_{x\to 0}\left(\frac{x\sin 2x - 3\cos 2x + 3}{x^2}\right) = 8$ **A1** **[7 Marks]**

3. a) Let $\frac{3}{(3r+1)(3r+4)} \equiv \frac{A}{3r+1} + \frac{B}{3r+4} \equiv \frac{A(3r+4) + B(3r+1)}{(3r+1)(3r+4)}$

So $3 \equiv A(3r+4) + B(3r+1)$

Substituting $r = -\frac{4}{3}$, $3 = -3B \therefore B = -1$

> **Alternatively:** You could also equate coefficients of r and equate constants on both sides of the equation to find the values of A and B

Substituting $r = -\frac{1}{3}$, $3 = 3A \therefore A = 1$ **M1**

So $\frac{3}{(3r+1)(3r+4)} \equiv \frac{1}{3r+1} - \frac{1}{3r+4}$ **A1**

b) $\sum_{r=1}^{30} \frac{3}{(3r+1)(3r+4)} = \sum_{r=1}^{30}\left(\frac{1}{3r+1} - \frac{1}{3r+4}\right)$

Hint: Notice that the expression is of the form $f(3r) - f(3r+3)$, which behaves in the same way as an expression of the form $f(r) - f(r+1)$, so the method of differences can be used

Let $r = 1: \frac{1}{4} - \frac{1}{7}$

$r = 2: \frac{1}{7} - \frac{1}{10}$

$r = 3: \frac{1}{10} - \frac{1}{13}$

$\vdots$

$r = 30: \frac{1}{91} - \frac{1}{94}$

Hint: Write out enough unsimplified terms of the sum to see which ones cancel and which ones remain

So the sum is $\frac{1}{4} - \frac{1}{94}$ **M1**

$= \frac{45}{188}$ **A1** **[4 Marks]**

4. a) $\frac{1}{2}e^{-2x^2} = \frac{1}{2}\left(1 + (-2x^2) + \frac{(-2x^2)^2}{2!} + \ldots\right)$ **M1**

$= \frac{1}{2}\left(1 - 2x^2 + \frac{4x^4}{2} - \ldots\right) = \frac{1}{2}(1 - 2x^2 + 2x^4 - \ldots)$ **M1**

$= \frac{1}{2} - x^2 + x^4 - \ldots$ **A1**

b) This expansion is valid for all values of x **A1**

c) We want to approximate $e^{-0.02}$, so let $e^{-2x^2} = e^{-0.02}$, i.e. $-2x^2 = -0.02$

So $x^2 = 0.01 \therefore x = \pm 0.1$ **A1**

So $e^{-2(\pm 0.1)^2} \approx 2\left(\frac{1}{2} - (\pm 0.1)^2 + (\pm 0.1)^4\right)$ **M1**

$= 0.9802$ **A1** **[7 Marks]**

5. a) $\ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \frac{(-2x)^5}{5} - \ldots$ **M1**

$= -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 - \frac{32}{5}x^5 - \ldots$ **A1**

This expansion is valid for $-1 < -2x \le 1$, i.e. $-\frac{1}{2} \le x < \frac{1}{2}$ **A1**

b) We want to approximate $\ln\left(\frac{2}{3}\right)$, so let $1 - 2x = \frac{2}{3}$

Technique: Find the value of x which makes the expression equal to the one we are trying to approximate, and then use that value of x in the series expansion, ignoring any higher powers of x

$\frac{1}{3} = 2x \therefore x = \frac{1}{6}$ and this is in the range $-\frac{1}{2} \le x < \frac{1}{2}$ **M1**

So $\ln\left(\frac{2}{3}\right) = \ln\left(1 - 2\times\frac{1}{6}\right) \approx -2\times\frac{1}{6} - 2\left(\frac{1}{6}\right)^2 - \frac{8}{3}\left(\frac{1}{6}\right)^3 - 4\left(\frac{1}{6}\right)^4 - \frac{32}{5}\left(\frac{1}{6}\right)^5$ **M1**

$= -0.405144\ldots = -0.405$ (3 d.p.) **A1** **[6 Marks]**

6. Let $f(x) = \ln(1 + e^{2x})$ so $f(0) = \ln(1 + e^{2\times 0}) = \ln(1+1) = \ln 2$ **M1**

Technique: Differentiate $f(x)$ twice to find the derivatives for use in the Maclaurin series expansion, and then substitute in the values of $f(0)$, $f'(0)$, and $f''(0)$

$f'(x) = \frac{2e^{2x}}{1+e^{2x}}$ so $f'(0) = \frac{2e^{2\times 0}}{1+e^{2\times 0}} = \frac{2\times 1}{1+1} = 1$ **M1**

$f''(x) = \frac{(1+e^{2x})\times 4e^{2x} - 2e^{2x}\times 2e^{2x}}{(1+e^{2x})^2} = \frac{4e^{2x} + 4e^{4x} - 4e^{4x}}{(1+e^{2x})^2} = \frac{4e^{2x}}{(1+e^{2x})^2}$ so $f''(0) = \frac{4e^{2\times 0}}{(1+e^{2\times 0})^2} = \frac{4\times 1}{(1+1)^2} = 1$ **M1**

Technique: Use the quotient rule to find $f''(x)$

So $\ln(1+e^{2x}) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

$= \ln 2 + x + \frac{x^2}{2} + \ldots$ **A1** **[4 Marks]**

7. a) $x^2+8x+16=(x+4)^2$ **M1**

$$=\left(4\left(1+\frac{x}{4}\right)\right)^2 \text{ (i.e. } k=4) \textbf{ A1}$$

Technique: Factorise, and then take out a factor of 4

Alternatively: Expand $\left(k\left(1+\frac{x}{4}\right)\right)^2$ and compare coefficients

b) $\ln(x^2+8x+16)=\ln\left(4\left(1+\frac{x}{4}\right)\right)^2=2\ln\left(4\left(1+\frac{x}{4}\right)\right)$ **M1**

Technique: Use the power law of logarithms, $\log a^b = b\log a$

$$=2\ln 4+2\ln\left(1+\frac{x}{4}\right) \textbf{ M1}$$

Technique: Use the multiplication law of logarithms, $\log ab = \log a + \log b$

$$\ln\left(1+\frac{x}{4}\right)=\frac{x}{4}-\frac{\left(\frac{x}{4}\right)^2}{2}+\frac{\left(\frac{x}{4}\right)^3}{3}-\ldots \textbf{ M1}$$

So $\ln(x^2+8x+16)=2\ln 4+2\ln\left(1+\frac{x}{4}\right)=2\ln 4+\frac{x}{2}-\frac{x^2}{16}+\frac{x^3}{96}-\ldots$ (allow $\ln 16$ or $4\ln 2$ instead of $2\ln 4$) **A1**

This expansion is valid for $-1<\frac{x}{4}\le 1$, i.e. $-4<x\le 4$ **A1** **[7 Marks]**

8. $\frac{1}{1\times 4}+\frac{1}{2\times 5}+\frac{1}{3\times 6}+\ldots+\frac{1}{n(n+3)}=\sum_{r=1}^{n}\frac{1}{r(r+3)}$

Let $\frac{1}{r(r+3)}\equiv\frac{A}{r}+\frac{B}{r+3}\equiv\frac{A(r+3)+Br}{r(r+3)}$

Technique: Express the r^{th} term of the series in partial fractions, and notice that the expression is of the form $f(r) - f(r + 3)$, so the standard method of differences needs to be adapted to include more terms in order to see the pattern

So $1\equiv A(r+3)+Br$

Substituting $r=-3$, $1=-3B$ $\therefore B=-\frac{1}{3}$

Substituting $r=0$, $1=3A$ $\therefore A=\frac{1}{3}$ **M1**

So $\frac{1}{r(r+3)}\equiv\frac{1/3}{r}-\frac{1/3}{r+3}=\frac{1}{3}\left(\frac{1}{r}-\frac{1}{r+3}\right)$ **M1**

$\therefore\sum_{r=1}^{n}\frac{1}{r(r+3)}=\frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{r}-\frac{1}{r+3}\right)$

Let $r=1: \frac{1}{3}-\frac{1}{12}$

Hint: Look carefully at the pattern to see which terms will cancel and which terms will remain

$r=2: \frac{1}{6}-\frac{1}{15}$

$r=3: \frac{1}{9}-\frac{1}{18}$

$r=4: \frac{1}{12}-\frac{1}{21}$

⋮

$r=n-2: \frac{1}{3(n-2)}-\frac{1}{3(n+1)}$

$r=n-1: \frac{1}{3(n-1)}-\frac{1}{3(n+2)}$

$r=n: \frac{1}{3n}-\frac{1}{3(n+3)}$ **M1**

So the sum is $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}-\frac{1}{3(n+1)}-\frac{1}{3(n+2)}-\frac{1}{3(n+3)}$ **M1**

$=\frac{11}{18}-\frac{(n+2)(n+3)+(n+1)(n+3)+(n+1)(n+2)}{3(n+1)(n+2)(n+3)}$ **M1**

$=\frac{11}{18}-\frac{n^2+5n+6+n^2+4n+3+n^2+3n+2}{3(n+1)(n+2)(n+3)}$

$=\frac{11}{18}-\frac{3n^2+12n+11}{3(n+1)(n+2)(n+3)}$ (i.e. $a=3$, $b=12$, $c=11$) **A1** **[6 Marks]**

9. a) **Show that $e^{2x}\cos ax=1+2x+\frac{x^2}{2}(4-a^2)+\frac{x^3}{3}(4-3a^2)+\ldots$**

$e^{2x}=1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\ldots=1+2x+2x^2+\frac{4}{3}x^3+\ldots$ **M1**

$\cos ax=1-\frac{(ax)^2}{2!}+\ldots=1-\frac{a^2x^2}{2}+\ldots$ **M1**

So $e^{2x}\cos ax=\left(1+2x+2x^2+\frac{4}{3}x^3+\ldots\right)\left(1-\frac{a^2x^2}{2}+\ldots\right)$

$=1-\frac{a^2x^2}{2}+2x-a^2x^3+2x^2+\frac{4}{3}x^3+\ldots$ (ignoring terms in x^4 and higher powers) **M1**

$=1+2x+\frac{x^2}{2}(4-a^2)+\frac{x^3}{3}(4-3a^2)+\ldots$ **A1**

b) $\frac{6\sin bx}{x}=\frac{6}{x}\left(bx-\frac{(bx)^3}{3!}+\ldots\right)=6b-b^3x^2+\ldots$ **M1**

So $e^{2x}\cos ax+\frac{6\sin bx}{x}-2x-4=1+2x+\frac{x^2}{2}(4-a^2)+\frac{x^3}{3}(4-3a^2)+\ldots+6b-b^3x^2+\ldots-2x-4$

$=6b-3+x^2\left(2-\frac{a^2}{2}-b^3\right)+x^3\left(\frac{4}{3}-a^2\right)+\ldots$ **M1**

cx^3 is the first non-zero term of this expansion, so $6b-3=0$ $\therefore b=\frac{1}{2}$ **A1**

Technique: Group the terms with the same power of x, and equate all terms lower than x^3 with zero to find the constants

Also, the x^2 term is equal to zero, so $2-\frac{a^2}{2}-b^3=0$

Since $b=\frac{1}{2}$, we have $2-\frac{a^2}{2}-\frac{1}{8}=0$

So $\frac{15}{8}=\frac{a^2}{2}$ $\therefore a=\sqrt{\frac{15}{4}}=\frac{\sqrt{15}}{2}$ (since a is positive) **A1**

We know that $x^3\left(\frac{4}{3}-a^2\right)=cx^3$, so with $a=\frac{\sqrt{15}}{2}$, $\frac{4}{3}-\frac{15}{4}=c$

$\therefore c=-\frac{29}{12}$ **A1** **[9 Marks]**

TOTAL 56 MARKS

1. a) $\int \frac{1}{25+x^2}\,dx = \frac{1}{5}\arctan\left(\frac{x}{5}\right)+c$ **A1**

b) $\int \frac{1}{25+9x^2}\,dx = \int \frac{1}{9\left(25/9+x^2\right)}\,dx$ **M1**

> **Technique:** Write $25 + 9x^2$ in the form $k(a^2 + x^2)$ and use the standard result $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$

$= \frac{1}{9}\left(\frac{1}{\left(\frac{5}{3}\right)}\arctan\left(\frac{x}{\left(\frac{5}{3}\right)}\right)\right)+c$ **M1**

$= \frac{1}{9}\left(\frac{3}{5}\arctan\left(\frac{3x}{5}\right)\right)+c$ **M1**

$= \frac{1}{15}\arctan\left(\frac{3x}{5}\right)+c$ **A1** **[5 Marks]**

2. a) Let $x = \frac{1}{3}\sin\theta$, so $\frac{dx}{d\theta} = \frac{1}{3}\cos\theta \therefore dx = \frac{1}{3}\cos\theta\, d\theta$ **M1**

$\therefore \int \frac{1}{\sqrt{1-9x^2}}\,dx = \int \frac{1}{\sqrt{1-9\left(\frac{1}{3}\sin\theta\right)^2}} \times \left(\frac{1}{3}\cos\theta\right) d\theta$ **M1**

> **Technique:** Substitute for x and use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to simplify

$= \int \frac{\frac{1}{3}\cos\theta}{\sqrt{1-9\times\frac{1}{9}\sin^2\theta}}\,d\theta = \int \frac{\frac{1}{3}\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$ **M1**

$= \int \frac{\frac{1}{3}\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{\frac{1}{3}\cos\theta}{\cos\theta}\,d\theta = \int \frac{1}{3}\,d\theta = \frac{1}{3}\theta + c$ **M1**

> **Hint:** Remember to rewrite the answer in terms of x at the end

Since $x = \frac{1}{3}\sin\theta$, $3x = \sin\theta$ so $\theta = \arcsin(3x)$

$\therefore \int \frac{1}{\sqrt{1-9x^2}}\,dx = \frac{1}{3}\theta + c = \frac{1}{3}\arcsin(3x)+c$, i.e. $A = \frac{1}{3}$ and $B = 3$ **A1**

> **Technique:** The mean value of f(x) over the interval [a, b] is $\frac{1}{b-a}\int_a^b f(x)\,dx$

b) The mean value of $f(x)$ on $\left[0, \frac{1}{6}\right]$ is $\frac{1}{\frac{1}{6}-0}\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx = 6\left[\frac{1}{3}\arcsin(3x)\right]_0^{\frac{1}{6}}$ **M1**

$= 6\left(\frac{1}{3}\arcsin\left(\frac{1}{2}\right) - \frac{1}{3}\arcsin(0)\right)$ **M1**

$= 6\left(\frac{\pi}{18} - 0\right) = \frac{\pi}{3}$ **A1**

> **Alternatively:** Recalculate the mean by integrating $-f(x)$ from 0 to 1/6

c) Every value of $-f(x)$ is the negative of the corresponding value of $f(x)$

So the mean of $-f(x)$ on $\left[0, \frac{1}{6}\right]$ is the negative of the mean of $f(x)$, i.e. $-\frac{\pi}{3}$ **A1** **[9 Marks]**

3. a) The upper limit is infinite **B1**

b) The area of the shaded region is $-\int_0^{\infty} -2e^{-x}dx$ **M1**

> **Technique:** The shaded region lies under the x-axis, so the value of the integral will be negative

$= -\lim_{t\to\infty}\int_0^t -2e^{-x}dx = -\lim_{t\to\infty}\left[2e^{-x}\right]_0^t$ **M1**

$= -\lim_{t\to\infty}\left(2e^{-t} - 2e^0\right) = -\lim_{t\to\infty}\left(2e^{-t} - 2\right)$ **M1**

$2e^{-t} \to 0$ as $t \to \infty$, so $-\lim_{t\to\infty}\left(2e^{-t} - 2\right) = -(-2) = 2$ **A1** **[5 Marks]**

4. a) Let $\frac{2x^2-3x+16}{(x-2)(x^2+5)} \equiv \frac{A}{x-2}+\frac{B}{x^2+5}$, so $A(x^2+5)+B(x-2) \equiv 2x^2-3x+16$

$\therefore Ax^2+5A+Bx-2B \equiv 2x^2-3x+16$

Equate x^2 coefficients: $A = 2$ **M1**

Equate x coefficients: $B = -3$ **M1**

So $\frac{2x^2-3x+16}{(x-2)(x^2+5)} = \frac{2}{x-2}-\frac{3}{x^2+5}$, i.e. $A = 2, B = -3$ **A1**

b) **Show that $\int_2^6 f(x)\,\mathrm{d}x$ diverges**

$$\int_2^6 f(x)\,\mathrm{d}x = \int_2^6 \frac{2x^2-3x+16}{(x-2)(x^2+5)}\,\mathrm{d}x = \int_2^6 \frac{2}{x-2}\,\mathrm{d}x - \int_2^6 \frac{3}{x^2+5}\,\mathrm{d}x$$

Technique: Consider whether each integral converges, and notice that the first integral is undefined at the lower limit

$$= \lim_{t\to 2}\int_t^6 \frac{2}{x-2}\,\mathrm{d}x - \int_2^6 \frac{3}{x^2+5}\,\mathrm{d}x \text{ **M1**}$$

$$= \lim_{t\to 2}\left[2\ln|x-2|\right]_t^6 - \left[\frac{3}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right]_2^6 \text{ **M1A1**}$$

$\lim_{t\to 2}\left[2\ln|x-2|\right]_t^6 = 2\ln 4 - \lim_{t\to 2}\left(2\ln|t-2|\right)$ **M1**

Tip: If one integral is finite, it is enough to find that the other is divergent to establish that the whole integral is divergent

$2\ln|t-2| \to -\infty$ as $t \to 2$, so $\int_2^6 \frac{2x^2-3x+16}{(x-2)(x^2+5)}\,\mathrm{d}x$ diverges **A1** **[8 Marks]**

5. $y = \arccos\left(\frac{x}{2}\right)$, so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\times\left(-\frac{1}{\sqrt{1-\left(x/2\right)^2}}\right)$ **M1**

Alternatively: Rewrite this as $x = 2\cos y$, then differentiate x with respect to y, and then use the relation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$

$$= -\frac{1}{2\sqrt{1-x^2/4}} = -\frac{1}{\sqrt{4\left(1-x^2/4\right)}} = -\frac{1}{\sqrt{4-x^2}} \text{ **M1**}$$

When $x = 1$, $y = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ **M1**

Also, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{4-1^2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$ **M1**

Alternatively: Use $y - y_1 = m(x-x_1)$ with $y_1 = \frac{\pi}{3}$, $m = -\frac{\sqrt{3}}{3}$, $x_1 = 1$

To find the equation of the tangent, use $y = mx + c$ with $y = \frac{\pi}{3}$, $m = -\frac{\sqrt{3}}{3}$, and $x = 1$:

$\frac{\pi}{3} = -\frac{\sqrt{3}}{3}\times 1 + c \therefore c = \frac{\pi}{3}+\frac{\sqrt{3}}{3}$ **M1**

So $y = -\frac{\sqrt{3}}{3}x+\frac{\pi}{3}+\frac{\sqrt{3}}{3}$

$\therefore 3y = \pi + \sqrt{3}(1-x)$, i.e. $a = 3$ and $b = 1$ **A1** **[6 Marks]**

6. $x = \frac{1}{10}\sin t,\ y = \frac{1}{100}t^3 \therefore \frac{dy}{dt} = \frac{3}{100}t^2$

> **Technique:** If the curve with parametric equations $x = f(t)$, $y = g(t)$ is rotated 2π radians about the y-axis between $t = a$ and $t = b$, then the volume of the solid generated is given by $V = \pi\int_a^b x^2 \frac{dy}{dt}\,dt$

So $V = \pi\int_0^{\pi}\left(\frac{1}{10}\sin t\right)^2 \times \frac{3}{100}t^2\ dt = \pi\int_0^{\pi}\frac{3}{10\,000}t^2\sin^2 t\ dt$ **M1**

> **Technique:** Use the identity $\cos 2t \equiv 1 - 2\sin^2 t$

$= \pi\int_0^{\pi}\frac{3}{10\,000}t^2\left(\frac{1}{2}(1-\cos 2t)\right)\ dt = \pi\int_0^{\pi}\frac{3}{20\,000}t^2(1-\cos 2t)\ dt$

$= \frac{3\pi}{20\,000}\left(\int_0^{\pi}t^2\ dt - \int_0^{\pi}t^2\cos 2t\ dt\right)$ **M1**

> **Technique:** Use integration by parts twice on the second integral

For the second integral, let $u = t^2$ and $\frac{dv}{dt} = \cos 2t$, so $\frac{du}{dt} = 2t$ and $v = \frac{1}{2}\sin 2t$

$\therefore \frac{3\pi}{20\,000}\left(\int_0^{\pi}t^2\ dt - \int_0^{\pi}t^2\cos 2t\ dt\right) = \frac{3\pi}{20\,000}\left(\left[\frac{1}{3}t^3\right]_0^{\pi} - \left[\frac{1}{2}t^2\sin 2t\right]_0^{\pi} + \int_0^{\pi}t\sin 2t\ dt\right)$ **M1**

Now let $u = t$ and $\frac{dv}{dt} = \sin 2t$, so $\frac{du}{dt} = 1$ and $v = -\frac{1}{2}\cos 2t$

$\therefore \int_0^{\pi}t\sin 2t\ dt = \left[-\frac{1}{2}t\cos 2t\right]_0^{\pi} - \int_0^{\pi}\left(-\frac{1}{2}\cos 2t\right)\ dt$ **M1**

$= -\frac{1}{2}\pi\cos 2\pi + 0 - \left[-\frac{1}{4}\sin 2t\right]_0^{\pi}$

$= -\frac{1}{2}\pi + \frac{1}{4}\sin 2\pi - \frac{1}{4}\sin 0 = -\frac{1}{2}\pi$ **M1**

So $V = \frac{3\pi}{20\,000}\left(\left[\frac{1}{3}t^3\right]_0^{\pi} - \left[\frac{1}{2}t^2\sin 2t\right]_0^{\pi} + \int_0^{\pi}t\sin 2t\ dt\right) = \frac{3\pi}{20\,000}\left(\frac{1}{3}\pi^3 - 0 - \frac{1}{2}\pi^2\sin 2\pi + 0 - \frac{1}{2}\pi\right)$

$= \frac{3\pi}{20\,000}\left(\frac{1}{3}\pi^3 - \frac{1}{2}\pi\right)$ **M1**

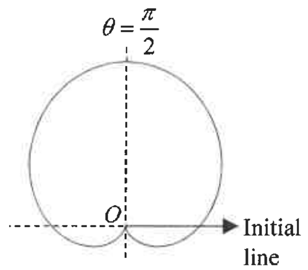
$= 0.00413023...\ \text{m}^3$

With 1 m^3, you can make $1 \div 0.00413023... = 242.117...$ **M1**

$= 242$ whole candles **A1** **[8 Marks]**

TOTAL 41 MARKS

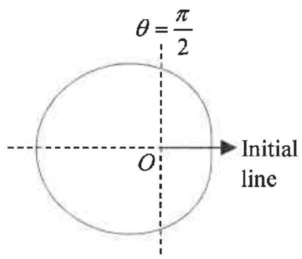
1. a) $r = 1 + \sin\theta$



For cardioid shape **A1**
For correct orientation **A1**

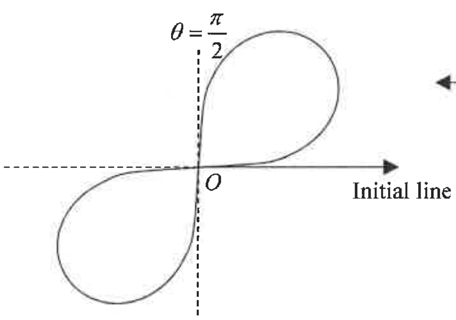
Hint: Curves of the form $r = p \pm q\cos\theta$ and $r = p \pm q\sin\theta$ look like deformed circles. They have a dimple if $q \le p < 2q$, as in part a), and are egg-shaped if $p \ge 2q$, as in part b). Their orientation depends on whether the sign is + or –, and whether the function is cos or sin.

b) $r = 2 - \cos\theta$



For egg-shaped curve **A1**
For correct orientation **A1**

c) $r^2 = 4\sin 2\theta$



Hint: Curves of the form $r^2 = a\cos k\theta$ and $r^2 = a\sin k\theta$ have k equally spaced loops coming from the origin in the regions where $\cos k\theta$ or $\sin k\theta$ is positive, respectively

For figure-of-eight shape **M1**
For correct orientation **A1**

[6 Marks]

2. $r = 2\cos 5\theta$

Each loop starts and ends where $r = 2\cos 5\theta = 0$, so where $5\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

So, for example, the loop which contains the initial line is formed by letting θ vary from $-\frac{\pi}{10}$ to $\frac{\pi}{10}$

So the area of one loop is:

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} (2\cos 5\theta)^2 \, d\theta \quad \textbf{M1}$$

$$= \frac{1}{2}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} 4\cos^2 5\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} (2\cos 10\theta + 2) \, d\theta \quad \textbf{M1}$$

$$= \frac{1}{2}\left[\frac{1}{5}\sin 10\theta + 2\theta\right]_{-\frac{\pi}{10}}^{\frac{\pi}{10}} \quad \textbf{A1}$$

$$= \frac{1}{2}\left(\frac{1}{5}\sin\pi + 2\times\frac{\pi}{10} - \frac{1}{5}\sin(-\pi) - 2\times\left(-\frac{\pi}{10}\right)\right)$$

$$= \frac{1}{5}\pi \quad \textbf{A1}$$

[4 Marks]

Technique: Determine the values of θ at which one of the loops starts and ends, and then integrate the curve between those limits. If you simply integrate $r^2 = (2\cos 5\theta)^2$ between $\theta = 0$ and $\theta = 2\pi$ and then divide by 5, you will get the wrong answer, since that will include parts of the curve where $r < 0$.

Tip: Rearrange the double angle formula for cosine, with 5θ instead of θ, in order to write $\cos^2 5\theta = \frac{1}{2}\cos 10\theta + \frac{1}{2}$

3. $r = 4 - 4\cos\theta,\ 0 < \theta < \pi$

a) $y = r\sin\theta = 4\sin\theta - 4\cos\theta\sin\theta$

$\frac{dy}{d\theta} = 4\cos\theta + 4\sin^2\theta - 4\cos^2\theta$ **A1**

$= 4\cos\theta + 4(1-\cos^2\theta) - 4\cos^2\theta$

$= 4 + 4\cos\theta - 8\cos^2\theta$

So $\frac{dy}{d\theta} = 0$ when $4 + 4\cos\theta - 8\cos^2\theta = 0$ **M1**

$\therefore (1-\cos\theta)(4+8\cos\theta) = 0$

So $\cos\theta = 1$ or $\cos\theta = -\frac{1}{2}$ **A1**

$\therefore \theta = \arccos(1) = 0$ or $\theta = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

but $0 < \theta < \pi$ so $\theta = \frac{2\pi}{3}$

and $r = 4 - 4\cos\theta = 4 - 4\times\left(-\frac{1}{2}\right) = 6$ **A1**

So $x = r\cos\theta = 6\times\left(-\frac{1}{2}\right) = -3$

and $y = r\sin\theta = 6\sin\frac{2\pi}{3} = 3\sqrt{3}$ **M1**

Alternatively: You can use the identity $\sin\theta \equiv \sqrt{1-\cos^2\theta}$ to write $\sin\theta$ in terms of $\cos\theta$ without having to work out θ itself

So the point has Cartesian coordinates $(-3, 3\sqrt{3})$ **A1**

b) $x = r\cos\theta = 4\cos\theta - 4\cos^2\theta$

$\frac{dx}{d\theta} = -4\sin\theta + 8\cos\theta\sin\theta$ **A1**

$= 4\sin\theta(2\cos\theta - 1)$

So $\frac{dx}{d\theta} = 0$ when $\sin\theta = 0$ or $2\cos\theta - 1 = 0$ **M1**

In the interval $0 < \theta < \pi$, $\sin\theta \neq 0$

$\therefore 2\cos\theta = 1$

$\therefore \cos\theta = \frac{1}{2}$ **A1**

and so $r = 4 - 4\cos\theta = 4 - 4\times\frac{1}{2} = 2$ **A1**

The tangent is perpendicular to the initial line, so has Cartesian equation $x = r\cos\theta = 2\times\frac{1}{2} = 1$ **M1**

and so the polar equation of the line is $r\cos\theta = 1$, hence $r = \sec\theta$ **A1** **[12 Marks]**

4. a) The curves $r = 1$ and $r = k\theta$ intersect at the point $(1, 2\pi)$, so $1 = k\times 2\pi$, and so $k = \frac{1}{2\pi}$ **A1**

b) Area of the circle is $\pi\times 1^2 = \pi$ **A1**

Area bounded by the curve $= \frac{1}{2}\int_0^{2\pi}\left(\frac{1}{2\pi}\theta\right)^2 d\theta$ **M1**

$= \frac{1}{2}\int_0^{2\pi}\frac{1}{4\pi^2}\theta^2\, d\theta$

$= \frac{1}{2}\left[\frac{1}{12\pi^2}\theta^3\right]_0^{2\pi}$ **A1**

$= \frac{1}{2}\left(\frac{1}{12\pi^2}\times 8\pi^3 - \frac{1}{12\pi^2}\times 0\right)$

$= \frac{1}{3}\pi$ **A1**

and so the shaded area is $\pi - \frac{1}{3}\pi = \frac{2}{3}\pi$ **A1** **[6 Marks]**

5. $r = a\cos\theta + \sec\theta$

$$\text{Area} = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (a\cos\theta + \sec\theta)^2 \, d\theta \quad \textbf{M1}$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(a^2\cos^2\theta + 2a\cos\theta\sec\theta + \sec^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{a^2}{2}\cos 2\theta + \frac{a^2}{2} + 2a + \sec^2\theta\right) d\theta \quad \textbf{M1}$$

$$= \frac{1}{2}\left[\frac{a^2}{4}\sin 2\theta + \left(\frac{a^2}{2} + 2a\right)\theta + \tan\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \quad \textbf{A1}$$

$$= \frac{1}{2}\left(\frac{a^2}{4}\sin\frac{\pi}{2} + \left(\frac{a^2}{2} + 2a\right)\times\frac{\pi}{4} + \tan\frac{\pi}{4} - \frac{a^2}{4}\sin\left(-\frac{\pi}{2}\right) - \left(\frac{a^2}{2} + 2a\right)\times\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{a^2}{4} + \left(\frac{a^2}{2} + 2a\right)\frac{\pi}{4} + 1 + \frac{a^2}{4} + \left(\frac{a^2}{2} + 2a\right)\frac{\pi}{4} + 1\right)$$

$$= \left(\frac{a^2}{4} + 1\right) + \left(\frac{a^2}{8} + \frac{a}{2}\right)\pi \quad \textbf{A1}$$

Tip: If *a*, *b*, *c* and *d* are rational and *a* + *b*π = *c* + *d*π, then *a* = *c* and *b* = *d* since π is irrational

We are told in the question that this area is equal to 17 + 12π, so $\frac{a^2}{4} + 1 = 17$ and $\frac{a^2}{8} + \frac{a}{2} = 12$ **M1**

The first of these equations gives $a^2 = 4\times(17-1) = 64$, so $a = \pm\sqrt{64} = \pm 8$

Trying these possibilities in the second equation, $a = -8$ gives $\frac{(-8)^2}{8} + \frac{-8}{2} = 4 \neq 12$, while $a = 8$ gives $\frac{8^2}{8} + \frac{8}{2} = 12$

So $a = 8$ **A1** **[6 Marks]**

6. a) The curves $r = 2 + 2\cos\theta$ and $r = 3$ intersect when $2 + 2\cos\theta = 3$ **M1**

So $2\cos\theta = 1$

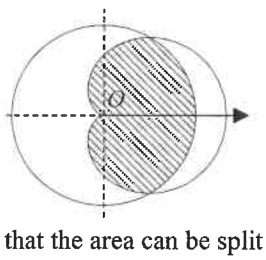
$\therefore \theta = \arccos\left(\frac{1}{2}\right)$ **A1**

We want $0 \leq \theta < 2\pi$, so $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

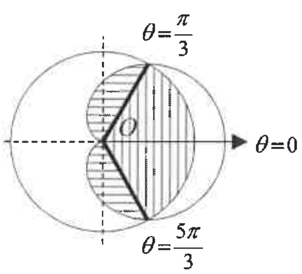
The points lie on the curve $r = 3$

So the points of intersection are $\left(3, \frac{\pi}{3}\right)$ and $\left(3, \frac{5\pi}{3}\right)$ **A1**

b) Sketching the area needed is useful here:



Note that the area can be split into two pieces. Part of the area is bounded by the curve $r = 3$ and a pair of radii, and part of it is bounded by the curve $r = 2 + 2\cos\theta$ and the same pair of radii, as sketched below:



Technique: When a region is contained within multiple curves, there will be values of θ where each curve forms the boundary. Determine which values of θ represent the change from one curve to another, and then integrate the correct curve between these values of θ.

The values of θ where the curve bounding the area changes over are the θ-values of the points of intersection, which from part a) are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3} = -\frac{\pi}{3}$. So the region bounded by $r = 3$ (the vertical stripes) is a sector of a circle with radius 3 containing an angle $\frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$.

So its area is $\frac{1}{2} \times 3^2 \times \frac{2\pi}{3} = 3\pi$ **M1A1**

Alternatively: You could also use integration to calculate this area

The area bounded by $r = 2 + 2\cos\theta$ (the horizontal stripes) is:

$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (2 + 2\cos\theta)^2 \, d\theta \quad \textbf{M1}$$

Alternatively: Because of the symmetry of the curve you can integrate between $\frac{\pi}{3}$ and π to find half the area then double the answer to obtain the required area

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(4 + 8\cos\theta + 4\cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \left(4 + 8\cos\theta + 2\cos 2\theta + 2\right) d\theta \quad \textbf{M1}$$

$$= \frac{1}{2}\left[6\theta + 8\sin\theta + \sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{5\pi}{3}} \quad \textbf{A1}$$

$$= \frac{1}{2}\left(6 \times \frac{5\pi}{3} + 8\sin\frac{5\pi}{3} + \sin\frac{10\pi}{3} - 6 \times \frac{\pi}{3} - 8\sin\frac{\pi}{3} - \sin\frac{2\pi}{3}\right)$$

$$= \frac{1}{2}\left(10\pi - 4\sqrt{3} - \frac{\sqrt{3}}{2} - 2\pi - 4\sqrt{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 4\pi - \frac{9\sqrt{3}}{2} \quad \textbf{A1}$$

So the total area of the region contained by both curves is:

$$3\pi + \left(4\pi - \frac{9\sqrt{3}}{2}\right) = 7\pi - \frac{9\sqrt{3}}{2} \quad \textbf{A1}$$

[10 Marks]

TOTAL 44 MARKS

1. a) $\frac{d}{dx}(\cosh 2x) = 2\sinh 2x$ **A1**

b) $\frac{d}{dx}\left(\frac{1}{4}\tanh 4x\right) = \frac{1}{4}\times 4\times \operatorname{sech}^2 4x = \operatorname{sech}^2 4x$ **A1**

c) $\frac{d}{dx}(x\sinh 3x) = 1\times \sinh 3x + x\times 3\cosh 3x$ **M1**

$= \sinh 3x + 3x\cosh 3x$ **A1**

d) $\frac{d}{dx}(\operatorname{arcosh} 3x) = \frac{1}{\sqrt{(3x)^2-1}}\times 3$ **M1**

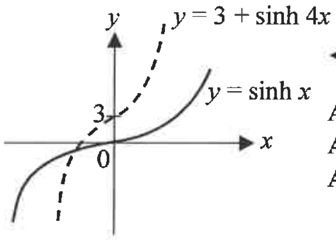
Technique: Use the standard result found in the formula book, but replace the *x* with 3*x* and use the chain rule

$= \frac{3}{\sqrt{9x^2-1}}$ **A1**   **[6 Marks]**

2. a) $\int \frac{3}{\sqrt{x^2+1}}\,dx = 3\int \frac{1}{\sqrt{x^2+1}}\,dx = 3\operatorname{arsinh} x + c$ **A1**

b) $\int \frac{4}{\sqrt{x^2-25}}\,dx = 4\int \frac{1}{\sqrt{x^2-25}}\,dx = 4\operatorname{arcosh}\left(\frac{x}{5}\right) + c$ **A1**   **[2 Marks]**

3.



Hint: *y* = sinh 4*x* is a stretch of *y* = sinh *x* by scale factor ¼ parallel to the *x*-direction. The + 3 means it is translated up by 3 units.

A1 correct shape for $y = \sinh x$
A1 correct shape for $y = 3 + \sinh 4x$
A1 $y = 3 + \sinh 4x$ has a y-intercept of 3
[3 Marks]

4. $5\cosh x + 3\sinh x = 5$

$5\left(\frac{e^x+e^{-x}}{2}\right) + 3\left(\frac{e^x-e^{-x}}{2}\right) = 5$

$\frac{5}{2}e^x + \frac{5}{2}e^{-x} + \frac{3}{2}e^x - \frac{3}{2}e^{-x} - 5 = 0$ **M1**

$4e^x - 5 + e^{-x} = 0$ **M1**

$4e^{2x} - 5e^x + 1 = 0$

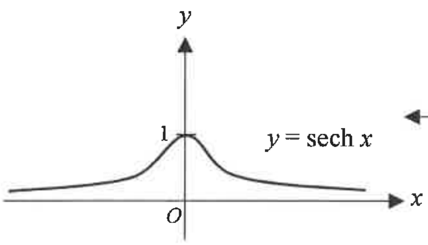
Technique: Rewrite the equation in exponential form then multiply by e^x to get a quadratic equation in e^x, and factorise to solve

$(4e^x - 1)(e^x - 1) = 0$ **M1**

So $e^x = \frac{1}{4}$ or $e^x = 1$ **M1**

So $x = \ln\frac{1}{4}$ or $x = \ln 1 = 0$ **A1A1**   **[6 Marks]**

5.



Tip: *y* = sech *x* is the graph of *y* = 1/ cosh *x*. To sketch it, it may help to think about the shape of *y* = cosh *x*.

A1 correct shape with maximum at $y = 1$
A1 asymptote at $y = 0$
[2 Marks]

6. a) **By writing $\operatorname{artanh} x = u$, show that $u = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$**

$u = \operatorname{artanh} x$, so $x = \tanh u = \frac{e^{2u}-1}{e^{2u}+1}$ where $|x|<1$ **M1**

Technique: Rearrange and rewrite the equation in exponential form, and then solve for u

$e^{2u} - 1 = x\left(e^{2u}+1\right)$

$e^{2u}(1-x) = 1+x$ **M1**

$e^{2u} = \frac{1+x}{1-x}$

$e^{u} = \sqrt{\frac{1+x}{1-x}}$ **M1**

Tip: Ignore the negative square root, as $e^u > 0$

$\therefore u = \ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$ **A1**

b) **Show that $y = \frac{3x-2}{3-2x}$**

$\operatorname{artanh} x - \operatorname{artanh} y = \ln\sqrt{5}$

So $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) = \ln\sqrt{5}$ **M1**

$\ln\sqrt{\frac{\left(\frac{1+x}{1-x}\right)}{\left(\frac{1+y}{1-y}\right)}} = \ln\sqrt{5}$ **M1**

Hint: Use the power law of logarithms, $a\log b = \log b^a$, and the subtraction law of logarithms, $\log a - \log b = \log(a/b)$

$\frac{\left(\frac{1+x}{1-x}\right)}{\left(\frac{1+y}{1-y}\right)} = 5 \therefore \frac{(1+x)(1-y)}{(1-x)(1+y)} = 5$ **M1**

$1 - y + x - xy = 5 + 5y - 5x - 5xy$

Technique: Rearrange and factorise the equation to isolate y

$6x - 4 = 6y - 4xy$ **M1**

$3x - 2 = y(3-2x)$ **M1**

$y = \frac{3x-2}{3-2x}$ **A1** **[10 Marks]**

7. **Use the substitution $x = \sqrt{\sinh u}$ to show that $\int \frac{2x}{\left(1+x^4\right)^{3/2}}\,dx = \frac{x^2}{\sqrt{1+x^4}} + c$**

$x = \sqrt{\sinh u} \therefore \frac{dx}{du} = \frac{1}{2} \times \frac{1}{\sqrt{\sinh u}} \times \cosh u$

So $dx = \frac{\cosh u}{2\sqrt{\sinh u}}\,du$ **M1**

$\therefore \int \frac{2x}{\left(1+x^4\right)^{3/2}}\,dx = \int \frac{2\sqrt{\sinh u}}{\left(1+\sinh^2 u\right)^{3/2}} \times \frac{\cosh u}{2\sqrt{\sinh u}}\,du$ **M1**

Technique: Use the identity $\cosh^2 u - \sinh^2 u \equiv 1$ to simplify

$= \int \frac{2\sqrt{\sinh u}\cosh u}{2\sqrt{\sinh u}\left(\cosh^2 u\right)^{3/2}}\,du = \int \frac{\cosh u}{\cosh^3 u}\,du = \int \frac{1}{\cosh^2 u}\,du = \int \operatorname{sech}^2 u\,du$ **M1**

$= \tanh u + c$ **M1**

$= \frac{\sinh u}{\cosh u} + c = \frac{\sinh u}{\sqrt{1+\sinh^2 u}} + c$ **M1**

Technique: Rewrite the answer in terms of x

$= \frac{x^2}{\sqrt{1+x^4}} + c$ **A1** **[6 Marks]**

8. Using the substitution $x = \frac{1}{8}(3 + 5\cosh u)$, $\frac{dx}{du} = \frac{5}{8}\sinh u$, so $dx = \frac{5}{8}\sinh u \, du$ **M1**

$$\therefore 4x^2 - 3x - 1 = 4\left(\frac{1}{8}(3+5\cosh u)\right)^2 - 3\left(\frac{1}{8}(3+5\cosh u)\right) - 1$$

$$= 4\left(\frac{9}{64} + \frac{15}{32}\cosh u + \frac{25}{64}\cosh^2 u\right) - 3\left(\frac{3}{8} + \frac{5}{8}\cosh u\right) - 1$$

$$= \frac{9}{16} + \frac{15}{8}\cosh u + \frac{25}{16}\cosh^2 u - \frac{9}{8} - \frac{15}{8}\cosh u - 1$$

$$= \frac{25}{16}\cosh^2 u - \frac{25}{16} \quad \textbf{M1}$$

$$= \frac{25}{16}\sinh^2 u \quad \textbf{M1}$$

Technique: Use the identity $\cosh^2 u - \sinh^2 u \equiv 1$ to rewrite $4x^2 - 3x - 1$ in terms of $\sinh u$

So $$\int \frac{1}{\sqrt{4x^2 - 3x - 1}}\,dx = \int \frac{1}{\sqrt{25/16 \sinh^2 u}} \times \frac{5}{8}\sinh u \, du = \int \frac{\frac{5}{8}\sinh u}{\frac{5}{4}\sinh u}\,du \quad \textbf{M1}$$

$$= \int \frac{1}{2}\,du = \frac{1}{2}u + c \quad \textbf{M1}$$

$x = \frac{1}{8}(3 + 5\cosh u) \therefore \cosh u = \frac{8x-3}{5}$ and so $u = \text{arcosh}\left(\frac{8x-3}{5}\right)$

So $$\int \frac{1}{\sqrt{4x^2 - 3x - 1}}\,dx = \frac{1}{2}\text{arcosh}\left(\frac{8x-3}{5}\right) + c \quad \textbf{A1}$$ **[6 Marks]**

9. Use the hyperbolic version of the addition formula, $R\sinh(x+\alpha) \equiv R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$

So $9\cosh x + 12\sinh x = R\sinh(x+\alpha)$

$$\equiv R\sinh x\cosh\alpha + R\cosh x\sinh\alpha$$

Equating terms, $R\cosh\alpha = 12$ and $R\sinh\alpha = 9$ **M1**

$$\therefore \frac{R\sinh\alpha}{R\cosh\alpha} = \tanh\alpha = \frac{9}{12}$$

Alternatively: Use the logarithmic form of artanh x to solve $\tanh\alpha = \frac{9}{12}$

$$\therefore \frac{e^{2\alpha} - 1}{e^{2\alpha} + 1} = \frac{9}{12} \quad \textbf{M1}$$

$$12e^{2\alpha} - 12 = 9e^{2\alpha} + 9$$

$$3e^{2\alpha} = 21$$

$$e^{2\alpha} = 7$$

$e^{\alpha} = \sqrt{7} \therefore \alpha = \ln\sqrt{7} = 0.972955... = 0.973$ (3 s.f.) **A1**

Hint: Ignore the negative root since e^{α} is positive for all real values of α

$$R^2(\cosh^2\alpha - \sinh^2\alpha) = R^2$$

$$\therefore (R\cosh\alpha)^2 - (R\sinh\alpha)^2 = R^2$$

$12^2 - 9^2 = 63 = R^2$ **M1**

$\therefore R = \sqrt{63} = 3\sqrt{7}$ **A1** **[5 Marks]**

10. $y = \text{arsinh}\left(\frac{x}{4}\right)$ so $\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{x}{4}\right)^2}} \times \frac{1}{4}$ **M1**

$$= \frac{\frac{1}{4}}{\sqrt{\frac{1}{16}(16+x^2)}} = \frac{\frac{1}{4}}{\frac{1}{4}\sqrt{16+x^2}} = \frac{1}{\sqrt{16+x^2}}$$

When $x = 3$, $\frac{dy}{dx} = \frac{1}{\sqrt{16+3^2}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$ **A1**

Also, when $x = 3$, $y = \text{arsinh}\left(\frac{3}{4}\right) = \ln\left(\frac{3}{4} + \sqrt{\left(\frac{3}{4}\right)^2 + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$ **M1**

So at $(3, \ln 2)$, equation of the tangent to the curve is $y = mx + c$, with $y = \ln 2$, $m = \frac{1}{5}$, $x = 3$

Alternatively: Use $y - y_1 = m(x - x_1)$ with $y_1 = \ln 2$, $m = \frac{1}{5}$, $x_1 = 3$

$\therefore \ln 2 = \frac{1}{5} \times 3 + c$ **M1**

$\therefore c = \ln 2 - \frac{3}{5}$

So $y = \frac{1}{5}x - \frac{3}{5} + \ln 2$ **A1** **[5 Marks]**

11. a) The curves $y = 7 + \sinh x$ and $y = 5\cosh x$ intersect where $5\cosh x = 7 + \sinh x$

$5\left(\frac{e^x + e^{-x}}{2}\right) = 7 + \frac{e^x - e^{-x}}{2}$ **M1**

$\frac{5}{2}e^x + \frac{5}{2}e^{-x} = 7 + \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$2e^x - 7 + 3e^{-x} = 0$

$2e^{2x} - 7e^x + 3 = 0$ **M1**

Using the quadratic formula with $a = 2$, $b = -7$, $c = 3$:

$e^x = \frac{7 \pm \sqrt{(-7)^2 - 4 \times 2 \times 3}}{2 \times 2}$ **M1**

$= \frac{7 \pm 5}{4}$ so $e^x = \frac{12}{4} = 3$ or $e^x = \frac{2}{4} = \frac{1}{2}$ **M1**

So $x = \ln 3$ or $x = \ln\frac{1}{2}$ **A1A1**

b) Area of $R = \int_{\ln\frac{1}{2}}^{\ln 3} (7 + \sinh x - 5\cosh x)\, dx$ **M1**

Alternatively: You can calculate the two areas under the curves separately, and then subtract one from the other

$= \left[7x + \cosh x - 5\sinh x\right]_{\ln\frac{1}{2}}^{\ln 3}$ **M1**

$= \left[7x + \frac{1}{2}e^x + \frac{1}{2}e^{-x} - \frac{5}{2}e^x + \frac{5}{2}e^{-x}\right]_{\ln\frac{1}{2}}^{\ln 3} = \left[7x - 2e^x + 3e^{-x}\right]_{\ln\frac{1}{2}}^{\ln 3}$ **M1**

$= \left(7\ln 3 - 2e^{\ln 3} + 3e^{-\ln 3}\right) - \left(7\ln\frac{1}{2} - 2e^{\ln\frac{1}{2}} + 3e^{-\ln\frac{1}{2}}\right)$ **M1**

$= 7\ln 3 - 2 \times 3 + 3 \times \frac{1}{3} - 7\ln\frac{1}{2} + 2 \times \frac{1}{2} - 3 \times \frac{1}{\frac{1}{2}}$

$= 7\ln\left(\frac{3}{\frac{1}{2}}\right) - 6 + 1 + 1 - 6$ **M1**

$= 7\ln 6 - 10$ **A1**

(so $a = 7$, $b = 6$, $c = -10$) **[12 Marks]**

TOTAL 63 MARKS

1. a) $y'' - 4y' - 5y = 0$

The auxiliary equation is $m^2 - 4m - 5 = 0$ **M1**

$\therefore (m+1)(m-5) = 0$

So $m = -1$ or $m = 5$ **A1**

So the general solution is $y = Ae^{-x} + Be^{5x}$ **A1**

b) $y'' - 4y' - 5y = 18e^{-x}$

The complementary function is $Ae^{-x} + Be^{5x}$ by part a)

> **Tip:** If the particular integral you would use is part of the complementary function, then multiply it by x and use that instead

The particular integral cannot be of the form λe^{-x} since that is part of the complementary function, so try the particular integral $y = \lambda xe^{-x}$, where $\lambda \in \mathbb{R}$

Then $y' = \lambda e^{-x} - \lambda xe^{-x}$ and $y'' = -2\lambda e^{-x} + \lambda xe^{-x}$ **A1**

So:

$$y'' - 4y' - 5y = (-2\lambda e^{-x} + \lambda xe^{-x}) - 4(\lambda e^{-x} - \lambda xe^{-x}) - 5\lambda xe^{-x} \quad \textbf{M1}$$

$$= -6\lambda e^{-x}$$

So $-6\lambda e^{-x} = 18e^{-x}$ **M1**

Hence $\lambda = \frac{18}{-6} = -3$ **A1**

So the particular integral is $-3xe^{-x}$

So the general solution is $y = -3xe^{-x} + Ae^{-x} + Be^{5x}$ **A1** **[8 Marks]**

2. $\frac{dy}{dx} - y\tan x = x$

The integrating factor is $e^{\int -\tan x \, dx} = e^{-\ln\sec x}$ **M1A1**

$= e^{\ln\cos x}$

$= \cos x$ **A1**

Multiplying the original equation by this gives:

$\cos x\frac{dy}{dx} - y\sin x = x\cos x$

$\therefore y\cos x = \int x\cos x \, dx$ **M1**

Use integration by parts to evaluate the integral

Let $u = x$, so $\frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \cos x$, so $v = \sin x$ **M1**

Then $\int x\cos x \, dx = x\sin x - \int \sin x \, dx$ **M1**

$= x\sin x + \cos x + c$ **A1**

So the original differential equation has solution

$y\cos x = x\sin x + \cos x + c$

$\therefore y = x\tan x + 1 + c\sec x$ (also accept $y = \frac{x\sin x + \cos x + c}{\cos x}$) **A1** **[8 Marks]**

3. $\ddot{x} = -4x$

a) The auxiliary equation is $m^2 + 4 = 0$ **M1**

So $m = \pm\sqrt{-4} = \pm 2i$ **A1**

Hence $x = A\cos 2t + B\sin 2t$ **A1**

We are told in the question that when $t = 0$, $x = 3$

$\therefore 3 = A\cos 0 + B\sin 0 = A$

So $A = 3$ and $x = 3\cos 2t + B\sin 2t$ **A1**

$\therefore \dot{x} = -6\sin 2t + 2B\cos 2t$ **A1**

We are told in the question that when $t = 0$, $\dot{x} = -8$

$\therefore -8 = -6\sin 0 + 2B\cos 0 = 2B$

So $B = -4$ **A1**

and so $x = 3\cos 2t - 4\sin 2t$ **A1**

b) Write $3\cos 2t - 4\sin 2t = R\cos(2t+\alpha)$

So $R\cos 2t\cos\alpha - R\sin 2t\sin\alpha = 3\cos 2t - 4\sin 2t$ **M1**

Then $R\cos\alpha = 3$ and $R\sin\alpha = 4$

So $R = \sqrt{3^2+4^2} = 5$ **M1**

So $x = 5\cos(2t+\alpha)$

So the maximum distance the particle reaches from O is 5 m **A1**

The period of the particle's motion is $\frac{2\pi}{2} = \pi$ seconds **A1**

Tip: You do not need to calculate α for this question

Hint: If displacement is given by the function $x = R\cos(\omega t+\alpha)$, then the maximum displacement is R and the period is $\frac{2\pi}{\omega}$

Alternatively: You can differentiate your expression for x to find its stationary points and thence its maximum and minimum values, but this would be much more work

[11 Marks]

4. $\frac{dx}{dt} = -y - 4t$

$\frac{dy}{dt} = 4x + 4y$

a) From the first differential equation, $y = -\frac{dx}{dt} - 4t$ **A1**

So $\frac{dy}{dt} = -\frac{d^2x}{dt^2} - 4$ **A1**

Substituting these into the second differential equation gives:

$-\frac{d^2x}{dt^2} - 4 = 4x - 4\frac{dx}{dt} - 16t$ **M1**

$\therefore \frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 16t - 4$ **(1)**

The complementary function of **(1)** is a solution of $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$

The auxiliary equation is $m^2 - 4m + 4 = 0$ **M1**

$\therefore (m-2)^2 = 0$ **M1**

So $m = 2$

So the complementary function is $x = (A+Bt)e^{2t}$ **A1**

The right-hand side of the differential equation **(1)** is a linear function of t, so try the particular integral $x = \lambda + \mu t$, where $\lambda, \mu \in \mathbb{R}$

Then $\frac{dx}{dt} = \mu$ and $\frac{d^2x}{dt^2} = 0$ **A1**

Substituting these into the differential equation **(1)** gives:

$0 - 4\mu + 4(\lambda + \mu t) = 16t - 4$ **M1**

Comparing coefficients of t gives: $4\mu = 16$, so $\mu = 4$

Comparing constant coefficients gives: $-16 + 4\lambda = -4$ so $\lambda = 3$ **M1**

Hence a particular integral is $3 + 4t$ **A1**

The general solution is the complementary function plus a particular integral, so:

$x = (A+Bt)e^{2t} + 3 + 4t$ **A1**

b) From part a) we know $y = -\frac{dx}{dt} - 4t$ and $x = (A+Bt)e^{2t} + 3 + 4t$

So $\frac{dx}{dt} = 2(A+Bt)e^{2t} + Be^{2t} + 4$ **A1**

Hence $y = -2(A+Bt)e^{2t} - Be^{2t} - 4 - 4t$ **A1**

We are told $x = 123$ when $t = 0$, so $123 = (A + B\times 0)e^{2\times 0} + 3 + 4\times 0 = A + 3$, so $A = 120$

We are also told that $y = 16$ when $t = 0$, so $16 = -2(120 + B\times 0)e^{2\times 0} - Be^{2\times 0} - 4 - 4\times 0 = -244 - B$, so $B = -260$ **A1**

and so $y = (520t + 20)e^{2t} - 4 - 4t$ **A1**

c) Exponential growth of a population is not realistic in the long term, so the number of bacteria is unlikely to grow exponentially for large t **B1**

[**OR** The rock cannot have a negative height, so once $x < 0$ the model becomes inappropriate **B1**] **[16 Marks]**

5. $\ddot{x}+2\dot{x}+5x=2\cos t$

The auxiliary equation is $m^2+2m+5=0$ **M1**

$\therefore (m+1)^2+4=0$

So $m=-1\pm 2\mathrm{i}$ **A1**

So the complementary function is $x=\mathrm{e}^{-t}\left(A\cos 2t+B\sin 2t\right)$ **A1**

The right-hand side of the differential equation is $2\cos t$, so for a particular integral try $x=\lambda\cos t+\mu\sin t$, where $\lambda, \mu\in\mathbb{R}$

Then $\dot{x}=-\lambda\sin t+\mu\cos t$ and $\ddot{x}=-\lambda\cos t-\mu\sin t$ **M1**

Substituting these into the differential equation gives:

$-\lambda\cos t-\mu\sin t+2\left(-\lambda\sin t+\mu\cos t\right)+5\left(\lambda\cos t+\mu\sin t\right)=2\cos t$ **M1**

After rearranging, this is:

$\left(4\lambda+2\mu\right)\cos t+\left(-2\lambda+4\mu\right)\sin t=2\cos t$

So equating, in turn, the coefficients of $\cos t$ and $\sin t$ gives:

$4\lambda+2\mu=2$

$-2\lambda+4\mu=0$

The second equation gives $\lambda=2\mu$. Substituting this into the first equation gives $10\mu=2$.

So $\mu=\frac{2}{10}=\frac{1}{5}$ and $\lambda=\frac{2}{5}$ **A1**

So a particular integral is $x=\frac{2}{5}\cos t+\frac{1}{5}\sin t$ **A1**

Hence the general solution is $x=\mathrm{e}^{-t}\left(A\cos 2t+B\sin 2t\right)+\frac{2}{5}\cos t+\frac{1}{5}\sin t$ **A1**

We are told that the rope starts at rest and that x is measured from its starting point

So at $t=0$, $x=0$ and $\dot{x}=0$

So $0=\mathrm{e}^0\left(A\cos 0+B\sin 0\right)+\frac{2}{5}\cos 0+\frac{1}{5}\sin 0=A+\frac{2}{5}$, and so $A=-\frac{2}{5}$ **A1**

Differentiating the general solution for x gives:

$\dot{x}=-\mathrm{e}^{-t}\left(A\cos 2t+B\sin 2t\right)+\mathrm{e}^{-t}\left(-2A\sin 2t+2B\cos 2t\right)-\frac{2}{5}\sin t+\frac{1}{5}\cos 2t$

Substituting in $t=0$, $\dot{x}=0$, $A=-\frac{2}{5}$ gives:

$0=-\mathrm{e}^0\left(-\frac{2}{5}\cos 0+B\sin 0\right)+\mathrm{e}^0\left(\frac{4}{5}\sin 0+2B\cos 0\right)-\frac{2}{5}\sin 0+\frac{1}{5}\cos 0=\frac{3}{5}+2B$

so that $B=-\frac{3}{10}$

and so $x=-\mathrm{e}^{-t}\left(\frac{2}{5}\cos 2t+\frac{3}{10}\sin 2t\right)+\frac{2}{5}\cos t+\frac{1}{5}\sin t$ **A1**

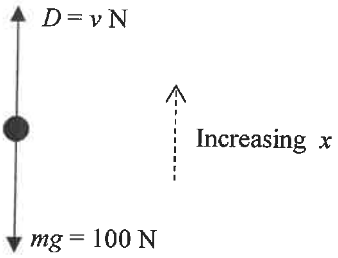
So at $t=3$ we have $x=-\mathrm{e}^{-3}\left(\frac{2}{5}\cos 6+\frac{3}{10}\sin 6\right)+\frac{2}{5}\cos 3+\frac{1}{5}\sin 3=-0.382721...$ **A1**

So $x<0$, and hence the centre of the rope is closer to Alphatron. Alphatron wins the tug of war. **A1** **[12 Marks]**

6. a) **Explain why $v=-\frac{dx}{dt}$, and hence show that the situation is modelled by the equation $\frac{d^2x}{dt^2}+2\frac{dx}{dt}=-10$**

v is the downward velocity of the meteor while x increases as you go upwards, hence $v=-\frac{dx}{dt}$ **A1**

A force diagram for the meteor is useful here:



Hint: Pay attention to your signs. x is the meteor's height above the ground, so increases as you go up, and hence $\frac{dx}{dt}$ is the meteor's upward velocity. v is the downward velocity of the meteor, so is the negative of $\frac{dx}{dt}$.

The resultant upward force is then $F=v-100$ N **A1**

So $F=-\frac{dx}{dt}-100$

By Newton's second law, $F=ma$. We know $a=\frac{d^2x}{dt^2}$, and we are told that $m=10$ kg, so:

$-\frac{dx}{dt}-100=10\frac{d^2x}{dt^2}$ **A1**

$\therefore 10\frac{d^2x}{dt^2}+\frac{dx}{dt}=-100$

b) The auxiliary equation of $10\frac{d^2x}{dt^2}+\frac{dx}{dt}=-100$ is $10m^2+m=0$ **M1**

$\therefore m(10m+1)=0$

So $m=0$ or $m=-\frac{1}{10}$ **A1**

So the complementary function is $x=A+Be^{-t/10}$ **A1**

The right-hand side of the differential equation is a constant, but a constant is part of the complementary function, so for a particular integral try $x=\lambda t$, where $\lambda\in\mathbb{R}$. Then $\frac{dx}{dt}=\lambda$ and $\frac{d^2x}{dt^2}=0$.

Substituting these into the differential equation gives: $0+\lambda=-100$ **M1**

So $\lambda=-100$

and hence a particular integral is $x=-100t$ **A1**

The general solution for x is then $x=-100t+A+Be^{-t/10}$ **A1**

We are told in the question that $\frac{dx}{dt}=-10$ when $t=0$

Differentiating the expression for x gives $\frac{dx}{dt}=-100-\frac{B}{10}e^{-t/10}$ **A1**

So $-10=-100-\frac{B}{10}e^{-0/10}$

$\therefore \frac{B}{10}=-100+10=-90$

So $B=-90\times10=-900$ **A1**

So $\frac{dx}{dt}=-100+90e^{-t/10}$

and so after 5 seconds the meteor has velocity $-100+90e^{-5/10}=-45.4122...=-45.4\text{ m s}^{-1}$ (1 d.p.)

and so its speed is 45.4 m s^{-1} (1 d.p.) **A1** **[12 Marks]**

TOTAL 67 MARKS