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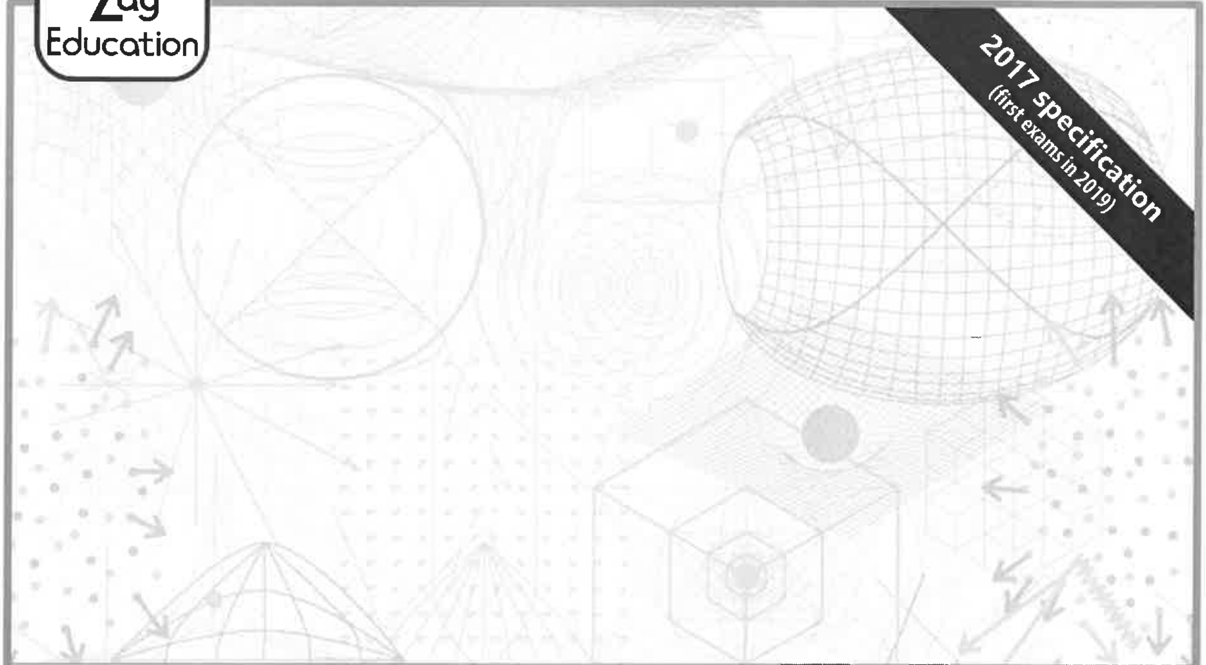

Topic Tests: Expert Tests – Set A

A Level Edexcel Further Mathematics: Core Pure Mathematics: Part 1#

#Every topic of AS (8FM0) Core Pure Mathematics

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POD 9428

Contents

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Resource ID & Name	9428 Topic Tests for AS / A Level Year 1 Edexcel Core Pure Further Mathematics: Expert Tests: Set A

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This resource is cross-referenced to the following textbook: the Pearson Education textbook *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* by Greg Attwood, Jack Barraclough, Ian Bettison, Lee Cope, Charles Garnet Cox, Daniel Goldberg, Alistair Macpherson, Bronwen Moran, Su Nicholson, Laurence Pateman, Joe Petran, Keith Pledger, Harry Smith, Geoff Staley and Dave Wilkins (ISBN 978-1292183336). ZigZag Education is not affiliated with Pearson Education in any way nor is this publication authorised by, associated with, sponsored by or endorsed by Pearson Education unless explicitly stated on the front cover of this publication.

Teacher's Introduction

Content

This pack contains 6 expert level topic tests for A Level Edexcel Further Mathematics: Core Pure Mathematics.

Each test comes with fully worked solutions, containing tips, hints and technique boxes to help students on a particular question. Answers should be given to three significant figures unless specified in the question.

These topic tests have been **fully cross-referenced** to the Pearson textbook, *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* (ISBN 978-1292183336), for your convenience (see reference sheet on page 2). Each test has been designed to reflect the specification fully.

About the expert tests

These **expert** tests have been designed to **prepare your students** for success in their exam. 25% of the marks come from questions similar in style to our fundamentals and challenge tests, giving all of your students a chance to show what they can do. The other 75% of the marks come from examination-style material, including compound and multistep questions that bring all parts of the topic together.

Timings

The recommended times for students to complete each test are given at the top of individual tests.

Calculator use

The effective use of a calculator is one of the objectives of the new specification and is encouraged for all the enclosed tests.

Also available from ZigZag Education

The perfect starting point for students of all abilities are our **fundamentals** tests. These isolate and test the core skills in each topic so that your students can show what they can do. They get a confidence boost and you can see at a glance where each student's weaknesses lie.

For students who are ready to go beyond the fundamentals, a complete set of **challenge** tests are available. 50% of the marks in these tests come from concepts covered in the fundamentals tests in order to reinforce learning and boost students' confidence, while the other 50% increases in difficulty and progresses the concepts covered.

For each collection of Set A tests we also offer a corresponding collection of Set B duplicated tests with the same styles of questions but different numbers. This allows for a variety of **flexible** uses including:

- **Test → Homework**: Students use test B as a homework to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Homework → Test**: Students revise as homework using test A before doing test B in class under test conditions.
- **Test → Classwork**: Students work through test B with teacher input to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Classwork → Test**: Students work through test A with teacher input, before checking their learning by completing test B under test conditions.

For total flexibility, the Set A and Set B tests of all three levels can be run on a rolling basis, using the fundamentals tests as starters, with a time interval between them, leaving one expert level test to use at the end of the course for topic revision.

Cross-referencing Grid

Topic	Edexcel spec. points	Subtopics	Chapter Reference - Edexcel Pearson textbook [ISBN: 9781292183336]
Complex Numbers	2.1–2.7	Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram	1, 2
Algebra and Functions	4.1–4.3	Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots	3, 4
Volumes of Revolution	5.1	Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes	5
Matrices	3.1–3.8	Matrices, matrix multiplication, determinants, inverting 2×2 and 3×3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation	6, 7
Proof by Induction	1.1	Proof by mathematical induction, proving divisibility results, proving statements involving matrices	8
Further Vectors	6.1–6.5	Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars	9

Complex Numbers – Test A (55 mins)

Fundamentals Challenge ✓ Expert

Subtopics: Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram

1. a) Given that $z_1 = 3+\mathrm{i}$ and $z_2 = 2-5\mathrm{i}$, find:

   i) $z_1 + 2z_2$ ii) $z_2 - 2z_1$ iii) $(z_2 - 2z_1)^*$ **[5]**

   b) Express $\dfrac{5-4\mathrm{i}}{3+6\mathrm{i}}$ in the form $a+b\mathrm{i}$, where a and b are rational numbers in their lowest terms. **[3]**

2. $z = \sqrt{3} - \mathrm{i}$

   a) Express z in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where θ is in radians and $-\pi < \theta \le \pi$ **[2]**

   b) Given that $|w| = 4$ and $\arg w = \dfrac{\pi}{4}$, find zw in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where θ is in radians and $-\pi < \theta \le \pi$ **[3]**

3. $f(z) = z^3 + z^2 + 20z + 78$

   a) **Verify that** $z = 1-5\mathrm{i}$ is a solution to $f(z) = 0$ **[4]**

   b) **Hence** show that $z^2 - 2z + 26$ is a factor of $f(z)$ **[2]**

   c) **Hence** solve $f(z) = 0$ **completely** **[2]**

   d) **Show** all three solutions on an Argand diagram **[3]**

4. Given that $|z - 4 - 5\mathrm{i}| = 4$,

   a) **sketch the locus** of z **[2]**

   b) find the **Cartesian equation** of the locus of z **[1]**

   c) find the **minimum** value of arg z in the interval $(-\pi, \pi)$ **to 3 significant figures** **[3]**

5. Find the **Cartesian equation** of the locus of the points representing $|z - 8\mathrm{i}| = |z - 4 + 4\mathrm{i}|$
   Give your answer in the form $ay = bx + c$, where a, b and c are integers **[5]**

6. $f(z) = z^4 - 7z^3 + 2z^2 + Az - 144$, where A is real. Given that $2-2\mathrm{i}$ is a root of $f(z) = 0$,

   a) **show that** $z^2 - 4z + 8$ is a factor of $f(z)$ **[2]**

   b) find the value of A and **hence** solve $f(z) = 0$ **completely** **[6]**

7. Find the range of values of θ in $-\dfrac{3\pi}{2} < \theta \le \dfrac{\pi}{2}$ for which $|z - 4 - 4\mathrm{i}| = 2$ and $\arg(z - 6 + 5\mathrm{i}) = \theta$ have **no common solutions**. Give your answers **exactly** where possible, or to **3 significant figures** otherwise. **[5]**

TOTAL 48 MARKS

Algebra and Functions – Test A (70 mins)

Subtopics: Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots

1. The equation $x^4 - 4x^3 + 6x - 20 = 0$ has roots α, β, γ and δ. **Without solving the equation**, find:
   a) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **[2]**
   b) $\alpha\beta\gamma\delta$ **[2]**
   c) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}$ **[2]**

   The quartic equation $20x^4 + Px^3 + 132x^2 - 94x + 21 = 0$ has roots $\frac{1}{\alpha}+1, \frac{1}{\beta}+1, \frac{1}{\gamma}+1$ and $\frac{1}{\delta}+1$
   d) Find the integer P **[2]**

2. a) Show that $\sum_{r=1}^{n}\left(2r^3+r\right)=\frac{1}{2}n(n+1)\left(n^2+n+1\right)$ **[2]**
   b) Hence, prove that there is **no** positive integer n that satisfies $\sum_{r=1}^{n}\left(2r^3+r\right)=\sum_{r=1}^{n^2} r$ **[4]**

3. The cubic equation $27x^3 + 81x^2 + 18x - 16 = 0$ has roots α, $\alpha - k$ and $\alpha + 2k$, where k is a constant. **Solve** the equation. You should show detailed reasoning for your answer. **[8]**

4. The **rational** numbers a and b satisfy $\sum_{r=1}^{8}(ar+b)=6$ and $\sum_{r=1}^{12}(ar+b)=17$
   a) Find a and b, giving your answers in their lowest terms. **[4]**
   b) For these values of a and b, find an expression for $\sum_{r=1}^{n}(ar+b)$ in terms of n. Factorise your answer as far as possible. **[2]**
   c) Hence find $\sum_{r=1}^{120}(ar+b)$ **[2]**

5. The cubic equation $36x^3 + px + 12 = 0$ has three **real** roots, α, β, and γ, where p is a constant.
   a) Find $\alpha + \beta + \gamma$ **[2]**
   b) Find $\alpha\beta\gamma$ **[2]**
   c) Given that $\gamma = 8\alpha$, find the roots of the equation **[5]**
   d) Hence find the value of p **[2]**

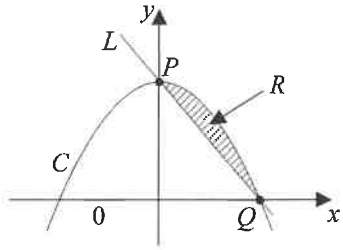
6. Find the positive integer n that satisfies $\sum_{r=1}^{n} r^2 = \sum_{r=1}^{7n}(2r-18)$ **[7]**

7. In this question, α, β, γ, and δ are the roots of the quartic equation $x^4 - 6x^3 - 7x^2 - 2x + 14 = 0$
   a) **Without solving the above equation**, find **integers** p and q such that the quartic equation $w^4 + pw^3 + 17w^2 - 10w + q = 0$ has roots $\alpha + 1, \beta + 1, \gamma + 1$ and $\delta + 1$ **[4]**
   b) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ **[2]**

TOTAL 54 MARKS

Subtopics: Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes

1. The curve C has equation $y = 25 - x^2$, and the line L has equation $5x + y = 25$. They intersect at the points P and Q, as sketched to the right.

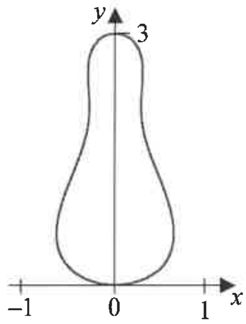


a) Find the **coordinates** of the points P and Q **[3]**

R is the shaded region bounded by C and L, as shown in the diagram.

b) Find the **exact** volume of the solid generated when R is rotated through 360° about the ***x*-axis** **[7]**

2. A designer models a giant bowling pin as the solid of revolution created by rotating the curve $21x^2 = 39y - 49y^2 + 21y^3 - 3y^4$ for $0 \le y \le 3$ (shown to the right) 360° about the ***y*-axis**, where each unit on the axes corresponds to 1 m.



a) Show that the volume of each bowling pin in m^3 given by this model is 2.09 m^3 to **2 decimal places**. **[3]**

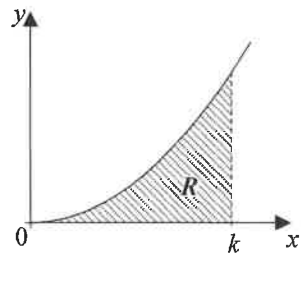
Each pin is usually cut from a **cylindrical** piece of wood that has height 3 m and radius 1 m. A mathematician suggests cutting the bowling pins out of a piece of wood in the shape of a **cone** with height 4 m and base radius 1 m.

b) If each cubic metre of wood costs 10p, how much cheaper is it to make one pin using the mathematician's suggestion? Give your answer **to the nearest penny**. **[3]**

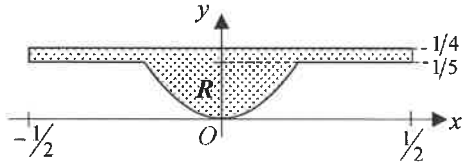
3. The curve C has equation $y = \lambda x^2 - 3$ for some real number $\lambda > 0$. When C is rotated 360° about the ***x*-axis** between $x = 0$ and $x = 2$, it generates a solid of revolution with volume 98π. When C is rotated 360° about the ***y*-axis** between $y = 1$ and $y = a$, where $a > 1$, it generates a solid of revolution with volume 2π. Find λ and a. **[9]**

4. The circle with centre the origin and radius r has equation $x^2 + y^2 = r^2$. By calculating a suitable volume of revolution, show, **by integration**, that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. **[3]**

5. The region R is bounded by the x-axis and the curve with equation $y = x^2$ between $x = 0$ and $x = k$, as shown to the right. Rotating R 360° about the y-axis generates a solid with volume $\frac{1}{2}\pi k^4$. Stephen says: 'As long as k is greater than 1, the volume of the solid generated when R is rotated 360° about the x-axis is greater than the volume of the solid generated when R is rotated 360° about the y-axis.' Show that Stephen is **incorrect**. **[5]**



6. A sports equipment manufacturer models a wobble board as the solid of revolution formed by rotating the region R 2π radians about the y-axis, where R is bounded by the curve $y = 5x^2$ for $0 \le y \le \frac{1}{5}$ and by the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ for $\frac{1}{5} \le y \le \frac{1}{4}$.



Each unit on the axes corresponds to 1 m.

a) Use this model to find the volume of the wobble board to **3 significant figures**. **[6]**

The manufacturer creates the wobble board, and then determines its volume to be 0.0577 m^3

b) Using this information and your answer to part a), comment on the suitability of the manufacturer's model. Explain your reasoning. **[1]**

TOTAL 40 MARKS

Subtopics: Matrices, matrix multiplication, determinants, inverting 2 × 2 and 3 × 3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation

1. a) For the matrices $\mathbf{A} = \begin{pmatrix} 3 & -2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 5 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$ find:

   i) $2\mathbf{A} - 3\mathbf{B}$ ii) $\mathbf{CD}$ iii) $\mathbf{D}^2$ iv) $\mathbf{C}^{-1}$ **[8]**

   b) Explain why you **cannot** find **DA** **[1]**

2. Given that $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & x \\ 2 & -2 \end{pmatrix}$, find:

   a) $\mathbf{A}^{\mathrm{T}}$ b) det **A** c) $\mathbf{B}^{-1}$ in terms of x **[6]**

   d) **AB**, given that the determinant of **B** is –4 **[4]**

3. Matrix $\mathbf{M} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$ and matrix $\mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Triangle T has vertices $A(p, q)$, $B\left(-1, -\sqrt{3}\right)$ and $C(2, r)$.

   a) **Describe fully** the transformations represented by the matrices **M** and **N** **[4]**

   b) **M** maps point A onto point A' with coordinates $\left(-\frac{3\sqrt{3}}{2}, \frac{1}{2}\right)$. Find the values of p and q. **[4]**

   c) Find $\mathbf{M}^3$ using your calculator. **[1]**

   d) Point B is mapped onto point B' by the transformation represented by $\mathbf{M}^3$. Describe fully the transformation represented by $\mathbf{M}^3$, and find the coordinates of B'. **[2]**

   e) Triangle T is mapped to triangle T' under matrix **N**. The area of T' is $18\sqrt{3}$. Given that $r > -\sqrt{3}$, find the value of r. **[3]**

4. The 2 × 2 matrix **R** represents a reflection in the line $y = -x$, and the matrix **S** represents a stretch with scale factor 3 parallel to the x-axis.

   a) Find the matrix **C**, where **C** = **RS** **[3]**

   b) Find $\mathbf{C}^{-1}$ and **describe fully** the sequence of transformations it represents **[2]**

5. a) Find the single transformation represented by the matrix $\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **[3]**

   b) Find the coordinates of the image of the point (2, 3, –1) after being transformed by matrix **M** **[1]**

   c) Given that **M** maps the point $(a^2, a - 7, 4a)$ onto the point $(-a^2, a + 3, 4a)$, find the value of a **[3]**

6. $\mathbf{M} = \begin{pmatrix} 3 & 4 & -1 \\ -2 & 6 & k \\ -4 & -1 & 2 \end{pmatrix}$ where k is a real constant.

   a) For which values of k is **M non-singular**? **[4]**

   b) Given that **M** is non-singular, find $\mathbf{M}^{-1}$ in terms of k **[4]**

7. A bookshop sells fiction, non-fiction and children's books. In 2016, the bookshop sold 1400 books. That year, the bookshop sold 20 more non-fiction books than children's books.
   In the same bookshop in 2017:
   - the sales of fiction books **increased** by 10%
   - the sales of non-fiction books **increased** by 10%
   - the sales of children's books **decreased** by 5%
   - **overall**, the sales of books **increased** by 7%

   Form and solve a matrix equation to find out how many of each type of book were sold in 2016. **[7]**

TOTAL 60 MARKS

Subtopics: Proof by mathematical induction, proving divisibility results, proving statements involving matrices

1. Prove by induction that $\sum_{r=1}^{n} 3^{r-1} = \frac{1}{2}\left(3^n - 1\right)$ for all positive integers n **[5]**

2. Let $\mathbf{A} = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}$. Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 2^n & n2^{n-2} \\ 0 & 2^n \end{pmatrix}$ for every natural number n. **[6]**

3. Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$ for all positive integers n **[6]**

4. Let $f(n) = 15^n - 7n^2 + 21n - 15$
   a) Show that $f(n+1) = f(n) + 14\left(15^n - n + 1\right)$ **[3]**
   b) Hence, or otherwise, prove by mathematical induction that $f(n)$ is divisible by 14 for every positive integer n **[4]**

5. Let $\mathbf{B} = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}$, where $a > 0$ is a constant.
   a) Prove by mathematical induction that $\mathbf{B}^n = \begin{pmatrix} a^n & na^{n-1}(a+1) \\ 0 & a^n \end{pmatrix}$ for all positive integers n **[6]**
   b) Hence find $\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}^5$ **[2]**

6. Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^2 + r} = \frac{n}{n+1}$ for all positive integers n **[6]**

7. Let $f(n) = 2^n + 3^{n-1}$. Colin is trying to prove that $f(n)$ is divisible by 3 for every natural number n. His attempt is as follows:

> When $n = 1$, $f(1) = 2^1 + 3^0 = 3$, so the statement is true when $n = 1$
> Assume the statement is true when $n = k$
> Now, $f(k+1) = 2^{k+1} + 3^k$
> $= 2 \times 2^k + 3^k$
> $= 2\left(f(k) - 3^{k-1}\right) + 3^k$
> $= 2\,f(k) - 2 \times 3^{k-1} + 3^k$
> $= 2\,f(k) + 3^{k-1}(3-2)$
> $= 2\,f(k) + 3^{k-1}$
> Since $f(k)$ is divisible by 3 (by assumption) and 3^{k-1} is divisible by 3, it follows that $f(k+1)$ is divisible by 3 too
> So by the principle of mathematical induction, $f(n)$ is divisible by 3 for every natural number n

   a) Identify the error in Colin's working. **[2]**
   b) Find a counterexample to the statement that Colin is trying to prove. **[1]**

TOTAL 41 MARKS

Further Vectors – Test A (80 mins)

Fundamentals Challenge ✓ **Expert**

Subtopics: Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars

1. The point $(1, p, 3)$ lies on the line with vector equation $\mathbf{r} = (6\mathbf{i} + 6\mathbf{j} + a\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$
Find the value of the constants p and a **[4]**

2. Three points are given by $A(-1, -7, 3)$, $B(1, 1, -1)$ and $C(2, -2, -2)$
a) Show that A, B and C **are not** collinear. **[4]**
The point D has coordinates $(3, 9, -5)$
b) Show that A, B, C and D **are** coplanar. **[5]**

3. The lines l_1 and l_2 are given by the vector equations $l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 10 \\ 12 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$
a) Show that the lines l_1 and l_2 intersect, and find the coordinates of their point of intersection **[5]**
b) Show that the **acute** angle between l_1 and l_2 is 81.8° to **1 decimal place** **[4]**
The line l_3 has Cartesian equation $\frac{x+1}{2} = \frac{y-4}{-2} = \frac{z-2}{-1}$. The lines l_1 and l_3 are skew.
c) Find the **shortest distance** between the lines l_1 and l_3 in the form $k\sqrt{2}$ for an **integer** k **[7]**

4. A submarine searches for a shipwreck. Relative to some fixed point O, the shipwreck is modelled as a point S with position vector $(6\mathbf{i} + 4\mathbf{j} - 5.2\mathbf{k})$ km. The submarine is modelled as a point travelling in a **straight line** from A, which has position vector $(16\mathbf{i} - 6\mathbf{j} - 4\mathbf{k})$ km, to B, which has position vector $(-12\mathbf{i} + 15\mathbf{j} - 4\mathbf{k})$ km. The submarine's sonar detects the shipwreck if the submarine passes within 2 km of the shipwreck.
a) Determine whether the submarine detects the shipwreck. **[8]**
b) Give one criticism of this model. **[1]**

5. The line L has equation $\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda(6\mathbf{i} + \mathbf{j} + \mathbf{k})$. The plane Π has equation $\mathbf{r}.(\mathbf{i} - \mathbf{k}) = -5$.
a) Find the coordinates of the **reflection** of the point $P = (1, 0, 1)$ in the plane Π **[6]**
L and Π intersect at the point $(-5, -1, 0)$
b) Show that the point P lies on the line L, and hence find a vector equation for L', the **reflection** of the line L in the plane Π **[3]**

6. A space probe is approaching Jupiter. Relative to some fixed point O, the probe's motion is modelled as a straight line that passes through the points $(49, 17, 8)$ and $(24, 12, 7)$, both of which lie outside Jupiter's atmosphere, and where the unit of distance is thousands of kilometres. The edge of the atmosphere of Jupiter is modelled as a plane with vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$. The probe will survive entry into Jupiter's atmosphere if the acute angle at which it enters is between 40° and 60°.
a) Find a vector equation for the plane modelling the edge of Jupiter's atmosphere in the form $\mathbf{r}.(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = d$, where a, b, c and d are **integers**. **[5]**
b) Determine whether the probe will survive entry into Jupiter's atmosphere. **[7]**
c) Give one criticism of the model. **[1]**

TOTAL 60 MARKS

1. a) $z_1 = 3+\mathrm{i},\ z_2 = 2-5\mathrm{i}$

i) $z_1 + 2z_2 = (3+\mathrm{i}) + 2(2-5\mathrm{i}) = 3+\mathrm{i}+4-10\mathrm{i}$ **M1**

$= 7-9\mathrm{i}$ **A1**

ii) $z_2 - 2z_1 = (2-5\mathrm{i}) - 2(3+\mathrm{i}) = 2-5\mathrm{i}-6-2\mathrm{i}$ **M1**

$= -4-7\mathrm{i}$ **A1**

iii) $(z_2 - 2z_1)^* = -4+7\mathrm{i}$ **A1**

Hint: z^* is the complex conjugate of $z = a + bi$, and is equal to $a - bi$

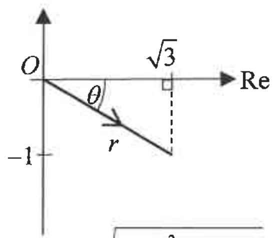
b) $\dfrac{5-4\mathrm{i}}{3+6\mathrm{i}} = \dfrac{5-4\mathrm{i}}{3+6\mathrm{i}} \times \dfrac{3-6\mathrm{i}}{3-6\mathrm{i}} = \dfrac{15-30\mathrm{i}-12\mathrm{i}+24\mathrm{i}^2}{9-18\mathrm{i}+18\mathrm{i}-36\mathrm{i}^2}$ **M1**

$= \dfrac{15-42\mathrm{i}-24}{9-(-36)} = \dfrac{-9-42\mathrm{i}}{45}$ **M1**

$= -\dfrac{9}{45} - \dfrac{42}{45}\mathrm{i} = -\dfrac{1}{5} - \dfrac{14}{15}\mathrm{i}$ **A1** **[8 Marks]**

Technique: Multiply the numerator and the denominator by the complex conjugate of the denominator to remove the imaginary component from the denominator

2. a) $z = \sqrt{3} - \mathrm{i}$



Tip: You may be able to use your calculator to convert a complex number from one form into another

$r = |z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2,\quad \theta = \arg z = -\arctan\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$ **M1**

$\therefore z = 2\left(\cos\left(-\dfrac{\pi}{6}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{6}\right)\right)$ **A1**

b) $|w| = 4$ and $\arg w = \dfrac{\pi}{4}$, so $w = 4\left(\cos\dfrac{\pi}{4} + \mathrm{i}\sin\dfrac{\pi}{4}\right)$ **M1**

Technique: Once both complex numbers are in modulus-argument form, to multiply them you simply multiply the moduli and add the arguments together

$\therefore zw = \left(2\left(\cos\left(-\dfrac{\pi}{6}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{6}\right)\right)\right) \times \left(4\left(\cos\dfrac{\pi}{4} + \mathrm{i}\sin\dfrac{\pi}{4}\right)\right)$

$= (2\times 4)\left(\cos\left(-\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right)$ **M1**

$= 8\left(\cos\dfrac{\pi}{12} + \mathrm{i}\sin\dfrac{\pi}{12}\right)$ **A1** **[5 Marks]**

Alternatively: Find the modulus and the argument of z from part a), and then find $|zw|$ using $|z| \times |w|$ and $\arg(zw)$ using $\arg(z) + \arg(w)$

3. $f(z) = z^3 + z^2 + 20z + 78$

a) **Verify that $z = 1 - 5\mathrm{i}$ is a solution to $f(z) = 0$**

$(1-5\mathrm{i})^2 = 1-5\mathrm{i}-5\mathrm{i}+25\mathrm{i}^2 = 1-25-10\mathrm{i} = -24-10\mathrm{i}$ **M1**

$(1-5\mathrm{i})^3 = (1-5\mathrm{i})^2(1-5\mathrm{i}) = (-24-10\mathrm{i})(1-5\mathrm{i})$

$= -24+120\mathrm{i}-10\mathrm{i}+50\mathrm{i}^2 = -24-50+110\mathrm{i} = -74+110\mathrm{i}$ **M1**

So $f(1-5\mathrm{i}) = (-74+110\mathrm{i}) + (-24-10\mathrm{i}) + 20\times(1-5\mathrm{i}) + 78$ **M1**

$= -74+110\mathrm{i}-24-10\mathrm{i}+20-100\mathrm{i}+78 = 0$ **A1**

b) **Show that $z^2 - 2z + 26$ is a factor of $f(z)$**

$1 - 5\mathrm{i}$ is a solution to $f(z) = 0$, so $1 + 5\mathrm{i}$ is also a solution (since complex roots occur in conjugate pairs)

Therefore, $(z - 1 + 5\mathrm{i})$ and $(z - 1 - 5\mathrm{i})$ are both factors of $f(z)$ **M1**

So $((z-1)+5\mathrm{i})((z-1)-5\mathrm{i}) = (z-1)^2 - 25\mathrm{i}^2 = (z^2 - 2z + 1) + 25 = z^2 - 2z + 26$ is a factor of $f(z)$ **A1** c)

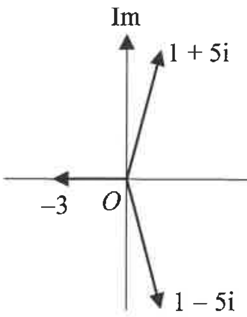
$$\begin{array}{r} z+\ 3 \\ z^2-2z+26\,\overline{)\,z^3+\ z^2+20z+78} \\ -\left(z^3-2z^2+26z\right) \\ \hline 3z^2-\ 6z+78 \\ -\left(3z^2-\ 6z+78\right) \\ \hline 0 \end{array}$$

Technique: $z^2 - 2z + 26$ is a factor of f(z) so f(z) can be written as $(z^2 - 2z + 26)(az + b)$. Use long division or inspection to find the linear factor.

So $f(z) = (z^2 - 2z + 26)(z + 3)$ **M1**

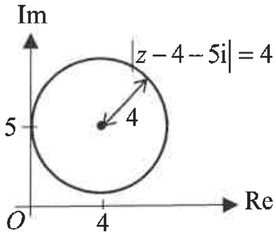
So the roots of $f(z) = 0$ are $1 - 5\text{i}$, $1 + 5\text{i}$, -3 **A1**

d)



A1A1A1 **[11 Marks]**

4. a)

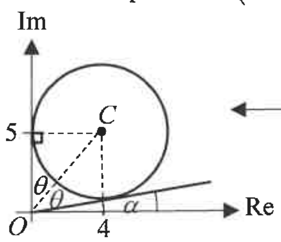


A1 circle with centre (4, 5)
A1 radius 4

Hint: The locus of $|z - a - b\text{i}| = r$ is a circle of radius r and centre (a, b)

b) Cartesian equation is $(x-4)^2+(y-5)^2=16$ **A1**

c)



Tip: Adding lines and reference points to your Argand diagram sketch will help you to find the angles involved in this question

The minimum value of arg z in the interval is α

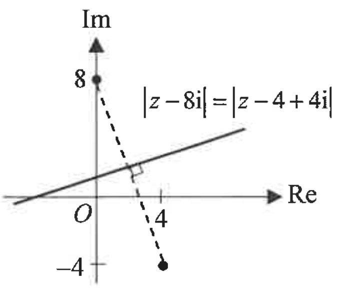
OC is the angle bisector of the angle made by the two tangents to the circle from O, so $\alpha = \dfrac{\pi}{2} - 2\theta$ **M1**

From diagram, $\theta = \arctan\left(\dfrac{4}{5}\right) \therefore \alpha = \dfrac{\pi}{2} - 2\arctan\left(\dfrac{4}{5}\right)$ **M1**

$= 0.221314... = 0.221$ (3 s.f.) **A1**

So the minimum value of arg z is 0.221 (3 s.f.) **[6 Marks]**

5.



Technique: The locus of $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line joining the points representing z_1 and z_2

The gradient of the line joining the points (0, 8) and (4, −4) is $\dfrac{-4-8}{4-0} = \dfrac{-12}{4} = -3$ **M1**

So the gradient of the perpendicular bisector is $\dfrac{-1}{-3} = \dfrac{1}{3}$ **M1**

The locus passes through the midpoint of (0, 8) and (4, –4), which is $\left(\frac{0+4}{2}, \frac{8-4}{2}\right) = (2, 2)$ **M1**

So the equation of the perpendicular bisector is $y = mx + c$; with $y = 2$, $m = \frac{1}{3}$, $x = 2$:

$2 = \frac{1}{3} \times 2 + c \therefore c = 2 - \frac{2}{3} = \frac{4}{3}$ **M1**

So the equation is $y = \frac{1}{3}x + \frac{4}{3}$ which rearranges to $3y = x + 4$ **A1** **[5 Marks]**

Alternatively: Use $y - y_1 = m(x - x_1)$ with $y_1 = 2$, $m = \frac{1}{3}$, $x_1 = 2$

6. $f(z) = z^4 - 7z^3 + 2z^2 + Az - 144$

a) **Show that $z^2 - 4z + 8$ is a factor of $f(z)$**

$2 - 2i$ is a root of $f(z) = 0$, so $2 + 2i$ is also a root (since complex roots occur in conjugate pairs)
Therefore, $(z - 2 - 2i)$ and $(z - 2 + 2i)$ are both factors of $f(z)$ **M1**
So $((z - 2) + 2i)((z - 2) - 2i) = (z - 2)^2 - 4i^2 = (z^2 - 4z + 4) + 4 = z^2 - 4z + 8$ is a factor of $f(z)$ **A1**

b)

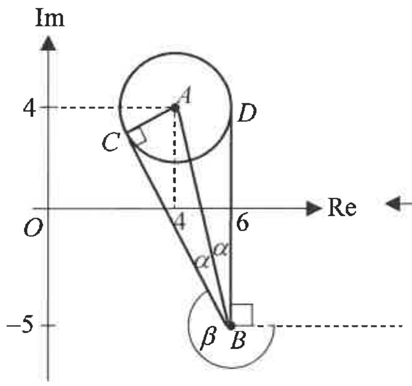
$$\begin{array}{r l}
 & z^2 - 3z - 18 \\
z^2 - 4z + 8 \,\big)\!\! & z^4 - 7z^3 + 2z^2 + Az - 144 \\
 & -(z^4 - 4z^3 + 8z^2) \\
 & -3z^3 - 6z^2 + Az - 144 \quad \mathbf{M1} \\
 & -(-3z^3 + 12z^2 - 24z) \\
 & -18z^2 + (A + 24)z - 144 \quad \mathbf{M1} \\
 & -(-18z^2 + 72z - 144) \\
 & (A - 48)z
\end{array}$$

Technique: $z^2 - 4z + 8$ is a factor of f(z) so f(z) can be written as $(z^2 - 4z + 8)(az^2 + bz + c)$. Use long division or inspection to find the second quadratic factor. The remainder is in terms of *A*, but it must be equal to 0, so use this to find *A*.

Alternatively: The question states that $2 - 2i$ is a root of $f(z) = 0$, so use the fact that $f(2 - 2i) = 0$ to work out the value of *A*

Since the remainder is 0, $A - 48 = 0 \therefore A = 48$ **A1**
So $f(z) = (z^2 - 4z + 8)(z^2 - 3z - 18)$ **M1**
$z^2 - 3z - 18$ factorises to $(z - 6)(z + 3)$ **M1**
So the roots of $f(z) = 0$ are $2 - 2i$, $2 + 2i$, 6, -3 **A1** **[8 Marks]**

7.



Tip: Sketch an Argand diagram: $|z - 4 - 4i| = 2$ is represented by a circle with centre *A*(4, 4) and radius 2, and $\arg(z - 6 + 5i) = \theta$ is represented by a half-line from *B*(6, –5) making an angle of θ with a horizontal line passing through (6, –5)

There are no common solutions when the value of θ is such that the half-line from (6, –5) does not touch or intersect the circle

So from the half-line *BD*, θ is in the interval $-\beta < \theta < \frac{\pi}{2}$ (since *D* lies on the circle) **A1**

AB is the angle bisector of the angle 2α made by *BC* and *BD*, the two tangents to the circle from *B*,

so $\beta = \frac{3\pi}{2} - 2\alpha$ **M1**

From diagram, length of $AB = \sqrt{(6-4)^2 + (-5-4)^2} = \sqrt{4+81} = \sqrt{85}$ and $AC = 2$ since it is a radius **M1**

Alternatively: Use the triangle *ABD* where $BD = 4 - (-5) = 9$ and $AD = 2$ since it is a radius. So $\alpha = \arctan\left(\frac{2}{9}\right)$.

So $\alpha = \arcsin\left(\frac{2}{\sqrt{85}}\right) \therefore \beta = \frac{3\pi}{2} - 2\arcsin\left(\frac{2}{\sqrt{85}}\right) = 4.27505... = 4.28$ (3 s.f.) **A1**

So the range of values where there are no common solutions is $\theta \in \left(-4.28, \frac{\pi}{2}\right)$ or $-4.28 < \theta < \frac{\pi}{2}$ **A1** **[5 Marks]**

TOTAL 48 MARKS

1. $x^4 - 4x^3 + 6x - 20 = 0$ has roots α, β, γ and δ

a) $$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$$
$$= -\frac{6}{1} \quad \textbf{M1}$$
$$= -6 \quad \textbf{A1}$$

b) $$\alpha\beta\gamma\delta = \frac{e}{a}$$
$$= \frac{-20}{1} \quad \textbf{M1}$$
$$= -20 \quad \textbf{A1}$$

c) $$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{\alpha\beta\gamma\delta}$$
$$= \frac{-6}{-20} \quad \textbf{M1}$$
$$= \frac{3}{10} \quad \textbf{A1}$$

d) $20x^4 + Px^3 + 132x^2 - 94x + 21 = 0$ has roots $\frac{1}{\alpha}+1$, $\frac{1}{\beta}+1$, $\frac{1}{\gamma}+1$ and $\frac{1}{\delta}+1$

$$-\frac{P}{20} = \left(\frac{1}{\alpha}+1\right) + \left(\frac{1}{\beta}+1\right) + \left(\frac{1}{\gamma}+1\right) + \left(\frac{1}{\delta}+1\right)$$
$$= \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}\right) + 4$$
$$= \frac{3}{10} + 4 \quad \textbf{M1}$$
$$= \frac{43}{10}$$

So $P = -20 \times \frac{43}{10} = -86$ **A1** **[8 Marks]**

2. a) **Show that $\sum_{r=1}^{n}(2r^3 + r) = \frac{1}{2}n(n+1)(n^2+n+1)$**

$$\sum_{r=1}^{n}(2r^3 + r) = 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r$$
$$= 2 \times \frac{1}{4}n^2(n+1)^2 + \frac{1}{2}n(n+1) \quad \textbf{M1}$$
$$= \frac{1}{2}n(n+1)[n(n+1)+1]$$
$$= \frac{1}{2}n(n+1)(n^2+n+1) \quad \textbf{A1}$$

Tip: Look for common factors and try to factorise the expression, where possible using factors that also appear in the given result. This is simpler than multiplying everything out and then attempting to factorise the resulting quartic polynomial.

b) **Prove that there is no positive integer n that satisfies $\sum_{r=1}^{n}(2r^3 + r) = \sum_{r=1}^{n^2} r$**

$$\sum_{r=1}^{n^2} r = \frac{1}{2}n^2(n^2+1) \quad \textbf{M1}$$

If $\sum_{r=1}^{n}(2r^3 + r) = \sum_{r=1}^{n^2} r$ then $\frac{1}{2}n(n+1)(n^2+n+1) = \frac{1}{2}n^2(n^2+1)$ **M1**

Divide both sides by $\frac{1}{2}n$ and expand to get:

$n^3+2n^2+2n+1=n^3+n$

which rearranges to become:

$2n^2+n+1=0$

The discriminant of this polynomial is $b^2-4ac=1^2-4\times2\times1=-7<0$ so there is no real solution n **M1**

Hence, in particular, there is no positive integer n such that $\sum_{r=1}^{n}\left(2r^3+r\right)=\sum_{r=1}^{n^2}r$ **A1 [6 Marks]**

3. $27x^3+81x^2+18x-16=0$ has roots α, $\alpha-k$, and $\alpha+2k$

$$\alpha+\alpha-k+\alpha+2k=-\frac{b}{a}$$

Alternatively: You could expand $27(x-\alpha)(x-\alpha+k)(x-\alpha-2k)$ then compare coefficients with $27x^3-81x^2+18x-16=0$

$$=-\frac{81}{27}\quad \textbf{M1}$$

$$=-3$$

So $3\alpha+k=-3$

This means $k=-3-3\alpha$

$$\alpha(\alpha-k)+\alpha(\alpha+2k)+(\alpha-k)(\alpha+2k)=\frac{c}{a}$$

$$=\frac{18}{27}\quad \textbf{M1}$$

$$=\frac{2}{3}$$

Also, $\alpha(\alpha-k)+\alpha(\alpha+2k)+(\alpha-k)(\alpha+2k)=\alpha^2-\alpha k+\alpha^2+2\alpha k+\alpha^2+\alpha k-2k^2$

$$=3\alpha^2+2\alpha k-2k^2$$

So $3\alpha^2+2\alpha k-2k^2=\frac{2}{3}$ **A1**

Substituting in $k=-3-3\alpha$ leads to:

$$3\alpha^2+2\alpha(-3-3\alpha)-2(-3-3\alpha)^2=3\alpha^2-6\alpha-6\alpha^2-18-36\alpha-18\alpha^2$$

$$=-21\alpha^2-42\alpha-18$$

So $-21\alpha^2-42\alpha-18=\frac{2}{3}$

Rearranging this gives: $21\alpha^2+42\alpha+\frac{56}{3}=0$ **M1**

and so $\alpha=\dfrac{-42\pm\sqrt{42^2-4\times21\times{}^{56}\!/_{3}}}{2\times21}$ **M1**

$$=\frac{-42\pm\sqrt{196}}{42}$$

So $\alpha=-\frac{4}{3}$ or $\alpha=-\frac{2}{3}$ **A1**

If $\alpha=-\frac{4}{3}$ then $k=-3-3\alpha=-3+4=1$

Then the three roots are $-\frac{4}{3}$, $-\frac{7}{3}$ and $\frac{2}{3}$

but then $\alpha(\alpha-k)(\alpha+2k)=\left(-\frac{4}{3}\right)\times\left(-\frac{7}{3}\right)\times\frac{2}{3}=\frac{56}{27}\neq-\frac{d}{a}=\frac{16}{27}$ **M1**

If $\alpha=-\frac{2}{3}$ then $k=-3-3\alpha=-3+2=-1$

Then the three roots are $-\frac{2}{3}$, $\frac{1}{3}$ and $-\frac{8}{3}$

and then $\alpha(\alpha-k)(\alpha+2k)=\left(-\frac{2}{3}\right)\times\frac{1}{3}\times\left(-\frac{8}{3}\right)=\frac{16}{27}=-\frac{d}{a}$

So the three roots are $-\frac{2}{3}, \frac{1}{3}$ and $-\frac{8}{3}$ **A1** **[8 Marks]**

4. a) $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + b\sum_{r=1}^{n} 1$

$= a \times \frac{1}{2}n(n+1) + bn$ **M1**

We are told in the question that $\sum_{r=1}^{8}(ar+b) = 6$ and $\sum_{r=1}^{12}(ar+b) = 17$, so

$\frac{a}{2} \times 8 \times (8+1) + b \times 8 = 6$

$\frac{a}{2} \times 12 \times (12+1) + b \times 12 = 17$ **M1**

Simplifying these gives the simultaneous equations:

$36a + 8b = 6$ **(1)**

$78a + 12b = 17$ **(2)**

$2 \times$**(2)** $- 3 \times$**(1)**:

$48a = 16$

So $a = \frac{16}{48} = \frac{1}{3}$ **A1**

Substituting this into **(1)** gives:

$36 \times \frac{1}{3} + 8b = 6$

$\therefore b = \frac{6 - \frac{1}{3} \times 36}{8} = -\frac{3}{4}$ **A1**

b) $\sum_{r=1}^{n}(ar+b) = \sum_{r=1}^{n}\left(\frac{1}{3}r - \frac{3}{4}\right)$

$= \frac{1}{3}\sum_{r=1}^{n} r - \frac{3}{4}\sum_{r=1}^{n} 1$

$= \frac{1}{3} \times \frac{1}{2}n(n+1) - \frac{3}{4}n$ **M1**

$= \frac{1}{12}n(2(n+1) - 9)$

$= \frac{1}{12}n(2n-7)$ **A1**

c) $\sum_{r=1}^{120}(ar+b) = \frac{1}{12} \times 120 \times (2 \times 120 - 7)$ **M1**

$= 2330$ **A1** **[8 Marks]**

5. $36x^3 + px + 12 = 0$ has roots α, β and γ

a) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= -\frac{0}{36}$ **M1**

$= 0$ **A1**

b) $\alpha\beta\gamma = -\frac{d}{a}$

$= -\frac{12}{36}$ **M1**

$= -\frac{1}{3}$ **A1**

c) We are told in the question that $\gamma = 8\alpha$. Substituting this into the two equations above gives:

$\alpha + \beta + \gamma = \alpha + \beta + 8\alpha = 9\alpha + \beta$, so $9\alpha + \beta = 0$ **(1)**

$\alpha\beta\gamma = \alpha\beta(8\alpha) = 8\alpha^2\beta$, so $8\alpha^2\beta = -\frac{1}{3}$ **(2)** **M1**

(1) gives $\beta = -9\alpha$. Substituting this into **(2)** gives:

$8\alpha^2 \times (-9\alpha) = -72\alpha^3 = -\frac{1}{3}$ **M1**

This simplifies to:

$\alpha^3 = \frac{1}{216}$

We are told in the question that the roots are real numbers, so $\alpha = \sqrt[3]{\frac{1}{216}} = \frac{1}{6}$ **A1**

Hence $\beta = -9\alpha = -9 \times \frac{1}{6} = -\frac{3}{2}$ **A1**

and $\gamma = 8\alpha = 8 \times \frac{1}{6} = \frac{4}{3}$ **A1**

d) $\frac{p}{36} = \alpha\beta + \alpha\gamma + \beta\gamma$

$= \frac{1}{6} \times \left(-\frac{3}{2}\right) + \frac{1}{6} \times \frac{4}{3} + \left(-\frac{3}{2}\right) \times \frac{4}{3}$ **M1**

$= -\frac{73}{36}$

and so $p = 36 \times \left(-\frac{73}{36}\right) = -73$ **A1** **[11 Marks]**

6. $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ **M1**

$\sum_{r=1}^{7n}(2r-18) = 2\sum_{r=1}^{7n} r - 18\sum_{r=1}^{7n} 1$

$= 2 \times \frac{1}{2} \times 7n(7n+1) - 18 \times 7n$ **M1**

$= 7n(7n-17)$

So these sums are equal when $\frac{1}{6}n(n+1)(2n+1) = 7n(7n-17)$ **M1**

Dividing through by $\frac{1}{6}n$ and multiplying out gives:

$2n^2 + 3n + 1 = 294n - 714$

$\therefore 2n^2 - 291n + 715 = 0$ **A1**

So $n = \frac{291 \pm \sqrt{(-291)^2 - 4 \times 2 \times 715}}{2 \times 2}$ **M1**

$= \frac{291 \pm \sqrt{78961}}{4}$

$= \frac{291 \pm 281}{4}$

so that $n = \frac{5}{2}$ or $n = 143$ **A1**

We require n to be a positive integer, so $n = 143$ **A1** **[7 Marks]**

7. $x^4 - 6x^3 - 7x^2 - 2x + 14 = 0$ has roots α, β, γ and δ

a) $w^4 + pw^3 + 17w^2 - 10w + q = 0$ has roots $\alpha + 1$, $\beta + 1$, $\gamma + 1$ and $\delta + 1$

$$-p = (\alpha+1)+(\beta+1)+(\gamma+1)+(\delta+1)$$
$$= (\alpha+\beta+\gamma+\delta)+4$$

Alternatively: Let $w = x + 1$, so that $x = w - 1$. Substituting this into the original equation gives a polynomial in w with roots $w = \alpha + 1, \beta + 1, \gamma + 1$ and $\delta + 1$.

We have

$$\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -\frac{-6}{1} = 6$$

So $-p = 6 + 4 = 10$ **M1**

Hence $p = -10$ **A1**

$$q = (\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$$
$$= \alpha\beta\gamma\delta$$
$$+\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta$$
$$+\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$$
$$+\alpha+\beta+\gamma+\delta$$
$$+1$$

We have

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a} = \frac{-7}{1} = -7$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a} = -\frac{-2}{a} = 2$$

$$\alpha\beta\gamma\delta = \frac{e}{a} = \frac{14}{1} = 14$$ **B1**

So $q = 14 + 2 - 7 + 6 + 1 = 16$ **A1**

b) $$\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$$

From part a), $\alpha + \beta + \gamma + \delta = 6$, and $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -7$, so:

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = 6^2 - 2\times(-7)$$ **M1**

$$= 50$$ **A1** **[6 Marks]**

TOTAL 54 MARKS

1. a) C has equation $y = 25 - x^2$, and L has equation $5x + y = 25$, i.e. $y = 25 - 5x$

The x-coordinates of P and Q are the solutions of $25 - x^2 = 25 - 5x$ **M1**

$\therefore x^2 - 5x = 0$

$\therefore x(x-5) = 0$

Alternatively: Notice that P lies on the y-axis and Q lies on the x-axis, and use this to find each of their other coordinates

So the x-coordinate of P is 0, and the x-coordinate of Q is 5 **A1**

The y-coordinate of P is then $25 - 0^2 = 25$, and the y-coordinate of Q is $25 - 5^2 = 0$ **A1**

So $P = (0, 25)$ and $Q = (5, 0)$

b) Let V_1 be the volume of the solid generated by rotating C 360° about the x-axis between $x = 0$ and $x = 5$

Let V_2 be the volume of the solid generated by rotating L 360° about the x-axis between $x = 0$ and $x = 5$

If V is the volume of the solid generated by rotating R about the x-axis, then $V = V_1 - V_2$

$$V_1 = \pi\int_0^5 \left(25 - x^2\right)^2 \mathrm{d}x \quad \textbf{M1}$$

$$= \pi\int_0^5 \left(625 - 50x^2 + x^4\right)\mathrm{d}x$$

$$= \pi\left[625x - \frac{50}{3}x^3 + \frac{1}{5}x^5\right]_0^5 \quad \textbf{A1}$$

$$= \pi\left(625\times5 - \frac{50}{3}\times125 + \frac{1}{5}\times3125 - 625\times0 + \frac{50}{3}\times0 - \frac{1}{5}\times0\right)$$

$$= \frac{5000}{3}\pi \quad \textbf{A1}$$

The solid generated by rotating L through 360° about the x-axis between $x = 0$ and $x = 5$ is a cone of radius 25 and height 5

$$\therefore V_2 = \frac{1}{3}\pi\times25^2\times5 \quad \textbf{M1}$$

Alternatively: You can use integration to find V_2

$$= \frac{3125}{3}\pi \quad \textbf{A1}$$

and so $V = V_1 - V_2$

$$= \frac{5000}{3}\pi - \frac{3125}{3}\pi \quad \textbf{M1}$$

$$= 625\pi \quad \textbf{A1}$$

[10 Marks]

2. a) **Show that the volume of each bowling pin in $\mathbf{m^3}$ given by this model is 2.09 $\mathbf{m^3}$ to 2 decimal places**

The curve has equation $21x^2 = 39y - 49y^2 + 21y^3 - 3y^4$, so $x^2 = \frac{13}{7}y - \frac{7}{3}y^2 + y^3 - \frac{1}{7}y^4$

$$V = \pi\int_0^3\left(\frac{13}{7}y - \frac{7}{3}y^2 + y^3 - \frac{1}{7}y^4\right)\mathrm{d}y \quad \textbf{M1}$$

$$= \pi\left[\frac{13}{14}y^2 - \frac{7}{9}y^3 + \frac{1}{4}y^4 - \frac{1}{35}y^5\right]_0^3 \quad \textbf{A1}$$

$$= \pi\left(\frac{13}{14}\times9 - \frac{7}{9}\times27 + \frac{1}{4}\times81 - \frac{1}{35}\times243 - \frac{13}{14}\times0 + \frac{7}{9}\times0 - \frac{1}{4}\times0 + \frac{1}{35}\times0\right)$$

$$= 2.08691\ldots$$

So the volume of the bowling pin is 2.09 m^3 (2 d.p.) **A1**

b) Each cylinder has a radius of $r = 1$ m and height $h = 3$ m, so its volume is:

$\pi r^2 h = \pi\times1^2\times3 = 3\pi\ \mathrm{m}^3$

At 10p per cubic metre this costs $3\pi\times10 = 94.2477\ldots$ pence **A1**

Each cone has a radius of $r = 1$ m and height $h = 4$ m, so its volume is:

$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\times1^2\times4 = \frac{4}{3}\pi\ \mathrm{m}^3$

At 10p per cubic metre, this costs $\frac{4}{3}\pi\times10 = 41.8879\ldots$ pence **A1**

And so, per pin, the cone is $94.2477\ldots - 41.8879\ldots = 52.3598\ldots = 52$p cheaper, to the nearest penny **A1**

[6 Marks]

3. C has equation $y = \lambda x^2 - 3$, so the volume of the solid generated by rotating C through 360° about the x-axis between $x = 0$ and $x = 2$ is:

$$V_x = \pi\int_0^2 \left(\lambda x^2 - 3\right)^2 \mathrm{d}x \quad \textbf{M1}$$

$$= \pi\int_0^2 \left(\lambda^2 x^4 - 6\lambda x^2 + 9\right)\mathrm{d}x$$

$$= \pi\left[\frac{\lambda^2}{5}x^5 - 2\lambda x^3 + 9x\right]_0^2 \quad \textbf{A1}$$

$$= \pi\left(\frac{\lambda^2}{5}\times 32 - 2\lambda\times 8 + 9\times 2 - \frac{\lambda^2}{5}\times 0 + 2\lambda\times 0 - 9\times 0\right)$$

$$= \pi\left(\frac{32}{5}\lambda^2 - 16\lambda + 18\right) \quad \textbf{A1}$$

We are told this volume is equal to 98π, so $\frac{32}{5}\lambda^2 - 16\lambda + 18 = 98$

So $\frac{32}{5}\lambda^2 - 16\lambda - 80 = 0$ **M1**

$$\therefore \lambda = \frac{16 \pm \sqrt{(-16)^2 - 4\times\left(\frac{32}{5}\right)\times(-80)}}{2\times\left(\frac{32}{5}\right)}$$

$$= \frac{16 \pm \sqrt{2304}}{\left(\frac{64}{5}\right)}$$

$$= \frac{16 \pm 48}{\left(\frac{64}{5}\right)}$$

and so $\lambda = -\frac{5}{2}$ or $\lambda = 5$, but we are told in the question that $\lambda > 0$, so $\lambda = 5$ **A1**

Since $\lambda = 5$, C has equation $y = 5x^2 - 3$, so $x^2 = \frac{y}{5} + \frac{3}{5}$. So the volume of the solid generated by rotating C 360° about the y-axis between $y = 1$ and $y = a$ is:

$$V_y = \pi\int_1^a \left(\frac{y}{5} + \frac{3}{5}\right)\mathrm{d}y$$

$$= \pi\left[\frac{1}{10}y^2 + \frac{3}{5}y\right]_1^a \quad \textbf{A1}$$

$$= \pi\left(\frac{1}{10}a^2 + \frac{3}{5}a - \frac{1}{10}\times 1 - \frac{3}{5}\times 1\right)$$

$$= \pi\left(\frac{1}{10}a^2 + \frac{3}{5}a - \frac{7}{10}\right) \quad \textbf{A1}$$

We are told this volume is equal to 2π, so $\frac{1}{10}a^2 + \frac{3}{5}a - \frac{7}{10} = 2$

So $\frac{1}{10}a^2 + \frac{3}{5}a - \frac{27}{10} = 0$ **M1** ←

Alternatively: You can use the quadratic formula here

Multiplying through by 10 gives $a^2 + 6a - 27 = 0$

$\therefore (a + 9)(a - 3) = 0$

and so $a = -9$ or $a = 3$. We are told in the question that $a > 1$, so $a = 3$. **A1** **[9 Marks]**

4. **Show that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$**

We can form a sphere by rotating the curve $x^2 + y^2 = r^2$ about either axis between $-r$ and r
Rotating 360° about the x-axis using $y^2 = r^2 - x^2$:

$$V = \pi\int_{-r}^{r}\left(r^2 - x^2\right)\mathrm{d}x \quad \textbf{M1}$$

Alternatively: If you want to rotate about the y-axis, then just rearrange the formula to give $x^2 = r^2 - y^2$ and integrate with respect to y instead

$$= \pi\left[r^2x - \frac{1}{3}x^3\right]_{-r}^{r} \quad \textbf{A1}$$

$$= \pi\left(r^2 \times r - \frac{1}{3}r^3 - r^2 \times (-r) + \frac{1}{3}(-r)^3\right)$$

$$= \pi\left(r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3\right)$$

$$= \frac{4}{3}\pi r^3 \quad \textbf{A1}$$

[3 Marks]

5. **Show that Stephen is incorrect**

R is the region bounded by the curve $y = x^2$ and the x-axis between $x = 0$ and $x = k$
Rotating R 360° about the x-axis:

$$V = \pi\int_0^k \left(x^2\right)^2 \mathrm{d}x \quad \textbf{M1}$$

$$= \pi\int_0^k x^4 \,\mathrm{d}x$$

$$= \pi\left[\frac{1}{5}x^5\right]_0^k \quad \textbf{A1}$$

$$= \pi\left(\frac{1}{5}k^5 - \frac{1}{5}\times 0\right)$$

$$= \frac{1}{5}\pi k^5 \quad \textbf{A1}$$

The volume, W, of the solid generated by rotating R 360° about the y-axis is given in the question as $W = \frac{1}{2}\pi k^4$

Stephen claims $V > W$ whenever $k > 1$; however, if $V > W$ then:

$$\frac{1}{5}\pi k^5 > \frac{1}{2}\pi k^4 \quad \textbf{M1}$$

$$\therefore \frac{k^5}{k^4} > \frac{5\pi}{2\pi}$$

$$\therefore k > \frac{5}{2}$$

and so if $1 < k \le \frac{5}{2}$ then $V \le W$, contradicting Stephen's claim, e.g. when $k = 2$, $V = \frac{32}{5}\pi = 6.4\pi$ while $W = 8\pi$, and so $V < W$ **A1**

[5 Marks]

6. a) Let V_1 be the volume of the solid generated by rotating $x = \frac{1}{2}$ through 2π radians about the y-axis between $y = \frac{1}{5}$ and $y = \frac{1}{4}$

Let V_2 be the volume of the solid generated by rotating $y = 5x^2$ through 2π radians about the y-axis between $y = 0$ and $y = \frac{1}{5}$

Then the volume V generated by rotating R through 2π radians about the y-axis is $V = V_1 + V_2$

V_1 is the volume of a cylinder of radius $r = \frac{1}{2}$ and height $h = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

So $V_1 = \pi r^2 h$

$= \pi \times \left(\frac{1}{2}\right)^2 \times \frac{1}{20}$ **M1**

$= \frac{\pi}{80}$ **A1**

For V_2, $y = 5x^2$

$\therefore x^2 = \frac{1}{5}y$ **M1**

So:

$V_2 = \pi \int_0^{\frac{1}{5}} \frac{1}{5} y \, dy$ **M1**

$= \pi \left[\frac{1}{10} y^2\right]_0^{\frac{1}{5}}$

$= \pi \left(\frac{1}{10} \times \frac{1}{25} - \frac{1}{10} \times 0\right)$

$= \frac{\pi}{250}$ **A1**

and so, according to the model, the volume of the wobble board is:

$V_1 + V_2 = \frac{\pi}{80} + \frac{\pi}{250}$

$= \frac{33}{2000}\pi$

$= 0.0518362...$

$= 0.0518 \text{ m}^3$ (3 s.f.) **A1**

b) For example: The model predicts 0.0518 m^3 but the actual wobble board has a volume of 0.0577 m^3

This is a percentage error of $\frac{0.0577 - 0.0518}{0.0577} \times 100 = 10.2253...\%$, which is very inaccurate, and so the model is not appropriate

[Or any other suitable comment on the difference between the two volumes] **B1** **[7 Marks]**

TOTAL 40 MARKS

1. $\mathbf{A}=\begin{pmatrix}3 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 5\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}4 & 0\\1 & 2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}3 & -1\\1 & 3\end{pmatrix}$

Technique: Multiply each element in matrix **A** by 2 and each element in matrix **B** by 3, then subtract corresponding elements

a) i) $2\mathbf{A}-3\mathbf{B}=2\begin{pmatrix}3 & -2\end{pmatrix}-3\begin{pmatrix}1 & 5\end{pmatrix}=\begin{pmatrix}6 & -4\end{pmatrix}-\begin{pmatrix}3 & 15\end{pmatrix}=\begin{pmatrix}3 & -19\end{pmatrix}$ **A1**

ii) $\mathbf{CD}=\begin{pmatrix}4 & 0\\1 & 2\end{pmatrix}\begin{pmatrix}3 & -1\\1 & 3\end{pmatrix}=\begin{pmatrix}4\times3+0\times1 & 4\times(-1)+0\times3\\1\times3+2\times1 & 1\times(-1)+2\times3\end{pmatrix}$ **M1**

$=\begin{pmatrix}12 & -4\\5 & 5\end{pmatrix}$ **A1**

Technique: To find each element in **CD**, find the sum of the elements in each row of **C** multiplied by the corresponding elements in each column of **D**

iii) $\mathbf{D}^2=\begin{pmatrix}3 & -1\\1 & 3\end{pmatrix}\begin{pmatrix}3 & -1\\1 & 3\end{pmatrix}=\begin{pmatrix}3\times3+(-1)\times1 & 3\times(-1)+(-1)\times3\\1\times3+3\times1 & 1\times(-1)+3\times3\end{pmatrix}$ **M1**

$=\begin{pmatrix}8 & -6\\6 & 8\end{pmatrix}$ **A1**

Technique: For a 2 × 2 matrix $\mathbf{M}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, $\mathbf{M}^{-1}=\frac{1}{\det\mathbf{M}}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$, where $\det\mathbf{M}=ad-bc\ (\neq 0)$

iv) $\det\mathbf{C}=\begin{vmatrix}4 & 0\\1 & 2\end{vmatrix}=4\times2-0\times1$ **M1**

$=8-0=8$ **A1**

So $\mathbf{C}^{-1}=\frac{1}{8}\begin{pmatrix}2 & 0\\-1 & 4\end{pmatrix}$ $\left(\text{or }\begin{pmatrix}\frac{1}{4} & 0\\-\frac{1}{8} & \frac{1}{2}\end{pmatrix}\right)$ **A1**

b) To multiply two matrices together, the number of columns in the first matrix needs to be equal to the number of rows in the second matrix. As **D** is a 2 × 2 matrix, **A** is a 1 × 2 matrix and 2 ≠ 1, we cannot calculate **DA**. **B1**
[Also allow: **D** and **A** are not multiplicatively conformable] **[9 Marks]**

Tip: You can remember this as 'the middle numbers must be equal for matrix multiplication to be possible', i.e. $\underset{2\times\underline{2}}{\mathbf{D}}\ \underset{\underline{1}\times2}{\mathbf{A}}$

2. $\mathbf{A}=\begin{pmatrix}2 & -3\\1 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}3 & x\\2 & -2\end{pmatrix}$

a) $\mathbf{A}^T=\begin{pmatrix}2 & 1\\-3 & 4\end{pmatrix}$ **A1**

Hint: $\mathbf{A}^T$ means **A** is transposed, i.e. rows and columns are interchanged

b) $\det\mathbf{A}=\begin{vmatrix}2 & -3\\1 & 4\end{vmatrix}=2\times4-(-3)\times1$ **M1**

$=8-(-3)=11$ **A1**

c) $\det\mathbf{B}=\begin{vmatrix}3 & x\\2 & -2\end{vmatrix}=3\times(-2)-2\times x$ **M1**

$=-6-2x$ **M1**

$\mathbf{B}^{-1}=\frac{1}{-6-2x}\begin{pmatrix}-2 & -x\\-2 & 3\end{pmatrix}$ **A1**

d) $\det\mathbf{B}=-4\ \therefore\ -6-2x=-4$ **M1**
$-2=2x\ \therefore\ x=-1$ **A1**

So $\mathbf{B}=\begin{pmatrix}3 & -1\\2 & -2\end{pmatrix}$

$\mathbf{AB}=\begin{pmatrix}2 & -3\\1 & 4\end{pmatrix}\begin{pmatrix}3 & -1\\2 & -2\end{pmatrix}$ **M1**

$=\begin{pmatrix}0 & 4\\11 & -9\end{pmatrix}$ **A1** **[10 Marks]**

Tip: Enter both matrices in your calculator and use it to multiply them together quickly or as a check

3. $\mathbf{M} = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

Tip: You can find rotation and reflection matrix transformations in the Edexcel formula book, but it is helpful if you can recognise more

a) **M** is a rotation by angle θ anticlockwise about the origin, so $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ **A1**

$\cos\theta = -\dfrac{\sqrt{3}}{2}$, $\sin\theta = -\dfrac{1}{2}$ so $\theta = 210°$ $\left(\text{or } \dfrac{7\pi}{6}\right)$ anticlockwise (or any equivalent angle) **A1**

N is an enlargement, centre (0, 0) **A1**
scale factor 2 **A1**

b) Under **M**, $A = (p, q)$ is mapped to $A' = \left(-\dfrac{3\sqrt{3}}{2}, \dfrac{1}{2}\right)$

So $\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\dfrac{3\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix}$

Alternatively: Set the matrix $\mathbf{M}\begin{pmatrix} p \\ q \end{pmatrix}$ equal to $\begin{pmatrix} -\frac{3\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$, then solve the resulting simultaneous equations to find p and q

det **M** = 1 and $\mathbf{M}^{-1} = \dfrac{1}{1}\begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$ **A1**

So $\begin{pmatrix} p \\ q \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} -\dfrac{3\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -\dfrac{3\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix}$ **M1**

$= \begin{pmatrix} 2 \\ -\sqrt{3} \end{pmatrix}$ so $p = 2$ and $q = -\sqrt{3}$ **A1A1**

c) $\mathbf{M}^3 = \begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{pmatrix}^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **A1**

Hint: $\mathbf{M}^3$ is the result of applying **M** three times, so it is a rotation of three times the angle rotated by **M**. Note that a rotation of 360° produces no effect, so a rotation of $\theta + 360°$ is equivalent to a rotation of θ.

d) $B = (-1, -\sqrt{3})$

$\mathbf{M}^3$ is a rotation by angle 270° anticlockwise about the origin (allow rotation by 630°) **A1**

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$ so $B' = (-\sqrt{3}, 1)$ **A1**

e) $A = (2, -\sqrt{3})$, $B = (-1, -\sqrt{3})$, $C = (2, r)$

A shape that has been enlarged by scale factor k has its area enlarged by scale factor k^2

So area of T' = area of $T \times 4$, i.e. area of $T = \dfrac{18\sqrt{3}}{4} = \dfrac{9\sqrt{3}}{2}$ **M1**

Using area of $T = \dfrac{1}{2} \times \text{base} \times \text{height} = \dfrac{1}{2} \times (2+1) \times (r + \sqrt{3})$:

Tip: A rough sketch of the known points confirms that the triangle T is right-angled

$\dfrac{9\sqrt{3}}{2} = \dfrac{3}{2}(r + \sqrt{3})$ **M1**

$3\sqrt{3} = r + \sqrt{3} \therefore r = 2\sqrt{3}$ **A1** **[14 Marks]**

4. a) Under a reflection in the line $y = -x$, the point (1, 0) is reflected to (0, −1) and the point (0, 1) is reflected to (−1, 0)

Technique: Consider what happens to the points (1, 0) and (0, 1) under this transformation

Hence, $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ **M1**

Under a stretch, scale factor 3 parallel to the x-axis, the point (1, 0) is stretched to (3, 0), but the point (0, 1) has x-coordinate zero, and so it does not move

Hence $\mathbf{S} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **M1**

So $\mathbf{C} = \mathbf{RS} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -3 & 0 \end{pmatrix}$ **A1**

b) $\mathbf{C}^{-1} = -\frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tfrac{1}{3} \\ -1 & 0 \end{pmatrix}$ **A1**

$\mathbf{C} = \mathbf{RS}$ so $\mathbf{C}^{-1} = (\mathbf{RS})^{-1} = \mathbf{S}^{-1}\mathbf{R}^{-1}$ so $\mathbf{C}^{-1}$ represents a reflection in the line $y = -x$ followed by a stretch with scale factor $\frac{1}{3}$ parallel to the x-axis **A1**

(or $\mathbf{C}^{-1}$ represents a stretch with scale factor $\frac{1}{3}$ parallel to the y-axis followed by a reflection in the line $y = -x$)

A1 **[5 Marks]**

Alternatively: Learn that a rotation by angle θ anticlockwise about the z-axis is given by $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5. $\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

a) The z-coordinate remains the same after the transformation given by **M** **M1**
So **M** is a rotation by angle θ anticlockwise about the z-axis **A1**
$\cos\theta = -1,\ \sin\theta = 0$ so $\theta = 180°$ or π (or any equivalent angle) **A1**

b) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ so the image has coordinates $(-2,\ -3,\ -1)$ **A1**

c) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a^2 \\ a-7 \\ 4a \end{pmatrix} = \begin{pmatrix} -a^2 \\ 7-a \\ 4a \end{pmatrix}$ **M1**

Technique: Find the image of the point after being transformed by **M**, and equate the coordinates to find the value of a

We are told that $(-a^2, 7 - a, 4a) = (-a^2, a + 3, 4a)$ $\therefore$ $7 - a = a + 3$ **M1**
$4 = 2a$ $\therefore$ $a = 2$ **A1** **[7 Marks]**

6. $\mathbf{M} = \begin{pmatrix} 3 & 4 & -1 \\ -2 & 6 & k \\ -4 & -1 & 2 \end{pmatrix}$

a) $\det \mathbf{M} = \begin{vmatrix} 3 & 4 & -1 \\ -2 & 6 & k \\ -4 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 6 & k \\ -1 & 2 \end{vmatrix} - 4\begin{vmatrix} -2 & k \\ -4 & 2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 6 \\ -4 & -1 \end{vmatrix}$ **M1**

$$= 3(12 + k) - 4(-4 + 4k) - (2 + 24)$$
$$= 3(12 + k) - 4(-4 + 4k) - 26 \quad \textbf{M1}$$
$$= 36 + 3k + 16 - 16k - 26$$
$$= 26 - 13k \quad \textbf{M1}$$

$26 - 13k = 0$ when $k = 2$ so **M** is non-singular for all $k \neq 2$ **A1**

Hint: A matrix is non-singular if $\det \mathbf{M} \neq 0$, and singular if $\det \mathbf{M} = 0$

b) To find $\mathbf{M}^{-1}$, first find the matrix of minors, $\mathbf{N}$:

$$\mathbf{N} = \begin{pmatrix} \begin{vmatrix} 6 & k \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} -2 & k \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 6 \\ -4 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -4 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -4 & -1 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ 6 & k \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ -2 & k \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ -2 & 6 \end{vmatrix} \end{pmatrix} \quad \mathbf{M1}$$

$$= \begin{pmatrix} 12+k & -4+4k & 26 \\ 7 & 2 & 13 \\ 4k+6 & 3k-2 & 26 \end{pmatrix} \quad \mathbf{M1}$$

Then find the transposed matrix of cofactors, $\mathbf{C}^{\mathrm{T}}$:

$$\mathbf{C} = \begin{pmatrix} 12+k & 4-4k & 26 \\ -7 & 2 & -13 \\ 4k+6 & 2-3k & 26 \end{pmatrix} \therefore \mathbf{C}^{\mathrm{T}} = \begin{pmatrix} 12+k & -7 & 4k+6 \\ 4-4k & 2 & 2-3k \\ 26 & -13 & 26 \end{pmatrix} \quad \mathbf{M1}$$

$$\text{So } \mathbf{M}^{-1} = \frac{1}{26-13k}\begin{pmatrix} 12+k & -7 & 4k+6 \\ 4-4k & 2 & 2-3k \\ 26 & -13 & 26 \end{pmatrix} \quad \mathbf{A1} \quad \textbf{[8 Marks]}$$

Technique: Divide the transposed matrix of cofactors by the determinant of the original matrix

7. Let x be the sales of fiction books in 2016, y be the sales of non-fiction books in 2016 and z be the sales of children's books in 2016

Then $x + y + z = 1400$ **M1**

Also, $y = z + 20$ **M1**

and $1.1x + 1.1y + 0.95z = 1400 \times 1.07 = 1498$ **M1**

Technique: Translate the information given in the question into equations, and rewrite those equations in matrix form

$$\text{So } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1.1 & 1.1 & 0.95 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1400 \\ 20 \\ 1498 \end{pmatrix} \quad \mathbf{A1}$$

Alternatively: The last row of the matrix could be written in terms of the percentage change in number of books sold, i.e.
$0.1x + 0.1y - 0.05z$
$= 0.07 \times 1400 = 98$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1.1 & 1.1 & 0.95 \end{pmatrix}^{-1}\begin{pmatrix} 1400 \\ 20 \\ 1498 \end{pmatrix} \quad \mathbf{M1}$$

Tip: You can use your calculator to find the inverse of the 3 × 3 matrix and save yourself some time

$$\left(\text{where } \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1.1 & 1.1 & 0.95 \end{pmatrix}^{-1} = -\frac{20}{3}\begin{pmatrix} 41/20 & 3/20 & -2 \\ -11/10 & -3/20 & 1 \\ -11/10 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -41/3 & -1 & 40/3 \\ 22/3 & 1 & -20/3 \\ 22/3 & 0 & -20/3 \end{pmatrix} \right)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 820 \\ 300 \\ 280 \end{pmatrix} \quad \mathbf{A1}$$

So 820 fiction books, 300 non-fiction books and 280 children's books were sold in 2016 **A1** **[7 Marks]**

TOTAL 60 MARKS

1. **Prove by induction that $\sum_{r=1}^{n} 3^{r-1} = \frac{1}{2}(3^n - 1)$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1} 3^{r-1} = 3^{1-1} = 1$ and $\frac{1}{2}(3^n - 1) = \frac{1}{2}(3^1 - 1) = 1$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k} 3^{r-1} = \frac{1}{2}(3^k - 1)$ **M1**

When $n = k + 1$:

$$\begin{aligned}\sum_{r=1}^{k+1} 3^{r-1} &= \sum_{r=1}^{k} 3^{r-1} + 3^{k+1-1}\\ &= \frac{1}{2}(3^k - 1) + 3^k \quad \textbf{M1}\\ &= \frac{3}{2}\times 3^k - \frac{1}{2}\\ &= \frac{1}{2}\times 3^{k+1} - \frac{1}{2}\\ &= \frac{1}{2}(3^{k+1} - 1) \quad \textbf{A1}\end{aligned}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[5 Marks]**

2. **Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 2^n & n2^{n-2} \\ 0 & 2^n \end{pmatrix}$ for every natural number n**

$\mathbf{A} = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}$

When $n = 1$: left-hand side $= \mathbf{A}^1 = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}$ and right-hand side $= \begin{pmatrix} 2^1 & 1\times 2^{1-2} \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 2^k & k2^{k-2} \\ 0 & 2^k \end{pmatrix}$ **M1**

$$\begin{aligned}\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A}\\ &= \begin{pmatrix} 2^k & k2^{k-2} \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 2 & \frac{1}{2} \\ 0 & 2 \end{pmatrix} \quad \textbf{M1}\\ &= \begin{pmatrix} 2^k\times 2 + k2^{k-2}\times 0 & 2^k\times\frac{1}{2} + k2^{k-2}\times 2 \\ 0\times 2 + 2^k\times 0 & 0\times\frac{1}{2} + 2^k\times 2 \end{pmatrix} \quad \textbf{A1}\\ &= \begin{pmatrix} 2^{k+1} & 2^{k-1} + k2^{k-1} \\ 0 & 2^{k+1} \end{pmatrix}\\ &= \begin{pmatrix} 2^{k+1} & (k+1)2^{k-1} \\ 0 & 2^{k+1} \end{pmatrix}\\ &= \begin{pmatrix} 2^{k+1} & (k+1)2^{(k+1)-2} \\ 0 & 2^{k+1} \end{pmatrix} \quad \textbf{A1}\end{aligned}$$

Alternatively: You can write $\mathbf{A}^{k+1}$ as $\mathbf{A}^k \times \mathbf{A}$ or as $\mathbf{A} \times \mathbf{A}^k$. You will get the same result either way.

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for every natural number n **A1** **[6 Marks]**

3. **Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1} r^3 = 1^3 = 1$ and $\frac{1}{4}n^2(n+1)^2 = \frac{1}{4}\times 1^2 \times 2^2 = 1$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$ **M1**

When $n = k + 1$:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \quad \textbf{M1}$$

$$= \frac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right)$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) \quad \textbf{A1}$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \frac{1}{4}(k+1)\left((k+1)+1\right)^2 \quad \textbf{A1}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

4. $f(n) = 15^n - 7n^2 + 21n - 15$

a) **Show that $f(n + 1) = f(n) + 14(15^n - n + 1)$**

$$f(n+1) - f(n) = \left(15^{n+1} - 7(n+1)^2 + 21(n+1) - 15\right) - \left(15^n - 7n^2 + 21n - 15\right) \quad \textbf{M1}$$

$$= 15^{n+1} - 7n^2 - 14n - 7 + 21n + 21 - 15 - 15^n + 7n^2 - 21n + 15$$

$$= 15^{n+1} - 15^n - 14n + 14$$

$$= 15^n(15-1) - 14n + 14$$

$$= 14(15^n - n + 1) \quad \textbf{A1}$$

So $f(n+1) = f(n) + 14(15^n - n + 1)$ **A1**

b) **Prove by mathematical induction that $f(n)$ is divisible by 14 for every positive integer n**

When $n = 1$: $f(1) = 15^1 - 7\times 1^2 + 21\times 1 - 15 = 14$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $f(k)$ is divisible by 14 **M1**

By part a), $f(k + 1) = f(k) + 14(15^k - k + 1)$

Alternatively: For the inductive step you can show that $f(k + 1)$ equals $15f(k) + 98k^2 - 308k + 224$. Since $f(k)$, 98, 308 and 224 are all divisible by 14, $f(k + 1)$ will be too.

Since $f(k)$ is divisible by 14 and $14(15^k - k + 1)$ is divisible by 14, their sum is divisible by 14 **M1**

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for every positive integer n **A1**

[7 Marks]

5. $\mathbf{B} = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}$

a) **Prove by mathematical induction that $\mathbf{B}^n = \begin{pmatrix} a^n & na^{n-1}(a+1) \\ 0 & a^n \end{pmatrix}$ for all positive integers n**

When $n = 1$: left-hand side $= \mathbf{B}^1 = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}$ and right-hand side $= \begin{pmatrix} a^1 & 1\times a^{1-1}(a+1) \\ 0 & a^1 \end{pmatrix} = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\mathbf{B}^k = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}^k = \begin{pmatrix} a^k & ka^{k-1}(a+1) \\ 0 & a^k \end{pmatrix}$ **M1**

$$\mathbf{B}^{k+1} = \mathbf{B}^k\mathbf{B}$$
$$= \begin{pmatrix} a^k & ka^{k-1}(a+1) \\ 0 & a^k \end{pmatrix}\begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix} \quad \textbf{M1}$$
$$= \begin{pmatrix} a^k \times a + ka^{k-1}(a+1)\times 0 & a^k \times (a+1) + ka^{k-1}(a+1)\times a \\ 0\times a + a^k \times 0 & 0\times(a+1)+a^k \times a \end{pmatrix} \quad \textbf{A1}$$
$$= \begin{pmatrix} a^{k+1} & (a^k + ka^k)(a+1) \\ 0 & a^{k+1} \end{pmatrix}$$
$$= \begin{pmatrix} a^{k+1} & (k+1)a^{(k+1)-1}(a+1) \\ 0 & a^{k+1} \end{pmatrix} \quad \textbf{A1}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1**

b) $\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}^5 = \begin{pmatrix} a & a+1 \\ 0 & a \end{pmatrix}^n$ with $a = 3$ and $n = 5$

So by part a),
$$\begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix}^5 = \begin{pmatrix} 3^5 & 5\times 3^4 \times 4 \\ 0 & 3^5 \end{pmatrix} \quad \textbf{M1}$$
$$= \begin{pmatrix} 243 & 1620 \\ 0 & 243 \end{pmatrix} \quad \textbf{A1}$$
[8 Marks]

6. **Prove by induction that $\sum_{r=1}^{n} \frac{1}{r^2+r} = \frac{n}{n+1}$ for all positive integers n**

Tip: You can use the method of differences to prove this statement more easily. However, the question tells you to use induction.

When $n = 1$: $\sum_{r=1}^{1} \frac{1}{r^2+r} = \frac{1}{1^2+1} = \frac{1}{2}$ and $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k} \frac{1}{r^2+r} = \frac{k}{k+1}$ **M1**

When $n = k + 1$:
$$\sum_{r=1}^{k+1} \frac{1}{r^2+r} = \sum_{r=1}^{k} \frac{1}{r^2+r} + \frac{1}{(k+1)^2+(k+1)}$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)^2+(k+1)} \quad \textbf{M1}$$

Tip: The denominator of the second fraction has a common factor of $(k + 1)$ so can be factorised as $(k + 1)[k + 1 + 1]$, which is $(k + 1)(k + 2)$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k(k+2)+1}{(k+1)(k+2)} \quad \textbf{M1}$$
$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$
$$= \frac{(k+1)^2}{(k+1)(k+2)}$$
$$= \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1} \quad \textbf{A1}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

7. a) During the inductive step, Colin has used the fact that 3^{k-1} is divisible by 3 to deduce that $f(k + 1)$ is divisible by 3 **B1** but when $k + 1 = 2$ this would require 3^0 to be divisible by 3 to be valid, but $3^0 = 1$ is not divisible by 3 **B1**

b) When $n = 2$, $f(2) = 2^2 + 3^1 = 4 + 3 = 7$, which is not divisible by 3 (or any other value of $n > 1$ as a counterexample) **A1** **[3 Marks]**

TOTAL 41 MARKS

Where they occur, λ, μ and ν are scalar parameters

1. $(1, p, 3)$ lies on the line with equation $\mathbf{r} = (6\mathbf{i} + 6\mathbf{j} + a\mathbf{k}) + \lambda(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

Using column vector notation, this means there is a value of λ satisfying:

$$\begin{pmatrix}1\\p\\3\end{pmatrix} = \begin{pmatrix}6\\6\\a\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix} = \begin{pmatrix}6+\lambda\\6-2\lambda\\a+3\lambda\end{pmatrix} \quad \textbf{M1}$$

After rearranging, this gives the simultaneous equations:

$\lambda = -5$ **(1)**

$p + 2\lambda = 6$ **(2)**

$a + 3\lambda = 3$ **(3)**

Equation **(1)** gives $\lambda = -5$ **M1**

Substituting this into equation **(2)** gives: $p + 2\times(-5) = 6$

$\therefore p = 6 + 2\times 5 = 16$ **A1**

and substituting $\lambda = -5$ into equation **(3)** gives: $a + 3\times(-5) = 3$

$\therefore a = 3 + 3\times 5 = 18$ **A1** **[4 Marks]**

2. $A(-1, -7, 3)$, $B(1, 1, -1)$, $C(2, -2, -2)$

a) **Show that *A*, *B* and *C* are not collinear**

Let L be the line that passes through A and B

Alternatively: You could also calculate the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and show that they are not multiples of each other

L has direction $\overrightarrow{AB} = \begin{pmatrix}1\\1\\-1\end{pmatrix} - \begin{pmatrix}-1\\-7\\3\end{pmatrix} = \begin{pmatrix}2\\8\\-4\end{pmatrix}$ **A1**

So L has vector equation $\mathbf{r} = \begin{pmatrix}-1\\-7\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\8\\-4\end{pmatrix}$ **M1**

If C lies on L, then there is a number λ such that:

$$\begin{pmatrix}2\\-2\\-2\end{pmatrix} = \begin{pmatrix}-1\\-7\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\8\\-4\end{pmatrix} = \begin{pmatrix}-1+2\lambda\\-7+8\lambda\\3-4\lambda\end{pmatrix}$$

This gives the equations:

$2 = -1 + 2\lambda$ **(1)**

$-2 = -7 + 8\lambda$ **(2)**

$-2 = 3 - 4\lambda$ **(3)** **M1**

Equation **(1)** has the solution $\lambda = \frac{3}{2}$

Equation **(2)** has the solution $\lambda = \frac{5}{8} \neq \frac{3}{2}$

So the equations are inconsistent. No suitable value of λ exists, which means C does not lie on L, and so A, B and C are not collinear. **A1**

b) **Show that *A*, *B*, *C* and *D* are coplanar**

Let Π be the plane that passes through A, B and C

So Π includes the point A and the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

A has position vector $\begin{pmatrix}-1\\-7\\3\end{pmatrix}$

From part a), $\overrightarrow{AB} = \begin{pmatrix}2\\8\\-4\end{pmatrix}$

and $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}$ **A1**

So Π has vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ -7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix}$ **A1**

D has position vector $\begin{pmatrix} 3 \\ 9 \\ -5 \end{pmatrix}$

If D lies in the plane Π, then there are numbers λ and μ such that:

$$\begin{pmatrix} 3 \\ 9 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -1+2\lambda+3\mu \\ -7+8\lambda+5\mu \\ 3-4\lambda-5\mu \end{pmatrix}$$

After rearranging, this gives the equations:

$2\lambda + 3\mu = 4$ **(1)**

$8\lambda + 5\mu = 16$ **(2)**

$4\lambda + 5\mu = 8$ **(3)** **M1**

$4\times$**(1)** – **(2)** gives:

$8\lambda + 12\mu - 8\lambda - 5\mu = 16 - 16$

$\therefore 7\mu = 0$

$\therefore \mu = 0$

Substituting this into **(1)**, say, gives:

$2\lambda + 3\times 0 = 4$

$\therefore 2\lambda = 4$

$\therefore \lambda = 2$ **A1**

Substituting $\lambda = 2$, $\mu = 0$ into equation **(3)** gives:

$4\times 2 + 5\times 0 = 8$

So the equations are consistent, and hence D lies in the plane Π **A1**

Hence A, B, C and D are coplanar **[9 Marks]**

Tip: You can use any two of the original three equations to find values for λ and μ, but you must then check whether these values also satisfy the third equation, which has not yet been used

3. $l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 7 \\ 1 \end{pmatrix}$, $l_2 : \mathbf{r} = \begin{pmatrix} 10 \\ 12 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

a) A general point on l_1 has position vector $\begin{pmatrix} 3+3\lambda \\ 7\lambda \\ -9+\lambda \end{pmatrix}$

A general point on l_2 has position vector $\begin{pmatrix} 10+\mu \\ 12-2\mu \\ -2+5\mu \end{pmatrix}$

So the lines meet when $\begin{pmatrix} 3+3\lambda \\ 7\lambda \\ -9+\lambda \end{pmatrix} = \begin{pmatrix} 10+\mu \\ 12-2\mu \\ -2+5\mu \end{pmatrix}$ **M1**

Rearranging, this gives the simultaneous equations:

$3\lambda - \mu = 7$ **(1)**

$7\lambda + 2\mu = 12$ **(2)**

$\lambda - 5\mu = 7$ **(3)**

Equation **(3)** gives $\lambda = 7 + 5\mu$. Substituting this into equation **(1)** gives:

$3(7+5\mu) - \mu = 7$ **M1**

$\therefore 14\mu = -14$

$\therefore \mu = -\dfrac{14}{14} = -1$

Hence $\lambda = 7 + 5\times(-1) = 2$ **A1**

Substituting $\lambda = 2$, $\mu = -1$ into equation **(2)** gives:

$7\times 2 + 2\times(-1) = 14 - 2 = 12$

So the equations are consistent, and the lines intersect when $\lambda = 2$, $\mu = -1$ **A1**

Using the equation for l_1, this corresponds to the point $(3 + 3\times 2, 7\times 2, -9 + 2) = (9, 14, -7)$ **A1**

b) **Show that the acute angle between l_1 and l_2 is 81.8° to 1 decimal place**

l_1 has direction $\begin{pmatrix}3\\7\\1\end{pmatrix}$ and l_2 has direction $\begin{pmatrix}1\\-2\\5\end{pmatrix}$

$$\left|\begin{pmatrix}3\\7\\1\end{pmatrix}\right| = \sqrt{3^2+7^2+1^2} = \sqrt{59}$$

$$\left|\begin{pmatrix}1\\-2\\5\end{pmatrix}\right| = \sqrt{1^2+(-2)^2+5^2} = \sqrt{30} \quad \textbf{M1}$$

and $\begin{pmatrix}3\\7\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix} = 3\times1+7\times(-2)+1\times5 = -6$ **A1**

Also, if the angle between the lines is θ, then:

$$\begin{pmatrix}3\\7\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix} = \left|\begin{pmatrix}3\\7\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\-2\\5\end{pmatrix}\right|\cos\theta$$

so that:

$$-6 = \sqrt{59}\sqrt{30}\cos\theta$$

So $\theta = \arccos\left(-\dfrac{6}{\sqrt{59}\sqrt{30}}\right) = 98.1991...°$ **A1**

We want the acute angle between the lines, which is $180° - 98.1991...° = 81.8008...° = 81.8°$ (1 d.p.) **A1**

c) l_3 has Cartesian equation $\dfrac{x+1}{2} = \dfrac{y-4}{-2} = \dfrac{z-2}{-1}$

So l_3 has vector equation $\mathbf{r} = \begin{pmatrix}-1\\4\\2\end{pmatrix} + v\begin{pmatrix}2\\-2\\-1\end{pmatrix}$ **B1**

A general point A on l_1 has position vector $\begin{pmatrix}3+3\lambda\\7\lambda\\-9+\lambda\end{pmatrix}$

A general point B on l_3 has position vector $\begin{pmatrix}-1+2v\\4-2v\\2-v\end{pmatrix}$

and so $\overrightarrow{AB} = \begin{pmatrix}-1+2v\\4-2v\\2-v\end{pmatrix} - \begin{pmatrix}3+3\lambda\\7\lambda\\-9+\lambda\end{pmatrix} = \begin{pmatrix}-4+2v-3\lambda\\4-2v-7\lambda\\11-v-\lambda\end{pmatrix}$ **M1**

If A and B are the closest possible points, then $\overrightarrow{AB}$ must be perpendicular to l_1, which has direction $\begin{pmatrix}3\\7\\1\end{pmatrix}$, so:

$$\begin{pmatrix}-4+2v-3\lambda\\4-2v-7\lambda\\11-v-\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\7\\1\end{pmatrix} = 0 \quad \textbf{M1}$$

$\therefore (-4+2v-3\lambda)\times3+(4-2v-7\lambda)\times7+(11-v-\lambda)\times1 = 0$ **M1**

$\therefore 27-9v-59\lambda = 0$ **(1)**

The vector $\overrightarrow{AB}$ must also be perpendicular to l_3, which has direction $\begin{pmatrix}2\\-2\\-1\end{pmatrix}$, so:

$$\begin{pmatrix}-4+2v-3\lambda\\4-2v-7\lambda\\11-v-\lambda\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\-1\end{pmatrix} = 0$$

$\therefore (-4+2v-3\lambda)\times2+(4-2v-7\lambda)\times(-2)+(11-v-\lambda)\times(-1) = 0$

$\therefore -27+9v+9\lambda = 0$ **(2)**

So we have the simultaneous equations:

$9v + 59\lambda = 27$ **(1)**

$9v + 9\lambda = 27$ **(2)**

(1) – **(2)** gives:

$9v + 59\lambda - 9v - 9\lambda = 27 - 27$ **M1**

$\therefore 50\lambda = 0$

So $\lambda = 0$

Substituting this into **(1)**, say, gives:

$9v + 59 \times 0 = 27$

$\therefore 9v = 27$

So $v = \dfrac{27}{9} = 3$

so that $\overrightarrow{AB} = \begin{pmatrix} -4+2v-3\lambda \\ 4-2v-7\lambda \\ 11-v-\lambda \end{pmatrix} = \begin{pmatrix} -4+6 \\ 4-6 \\ 11-3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 8 \end{pmatrix}$ **A1**

Hence $\left|\overrightarrow{AB}\right| = \sqrt{2^2 + (-2)^2 + 8^2} = \sqrt{72} = 6\sqrt{2}$

So the shortest distance between the two lines is $6\sqrt{2}$ (so $k = 6$) **A1** **[16 Marks]**

4. a) The shipwreck S has position vector $\overrightarrow{OS} = \begin{pmatrix} 6 \\ 4 \\ -5.2 \end{pmatrix}$

The submarine travels in a straight line L from A with position vector $\begin{pmatrix} 16 \\ -6 \\ -4 \end{pmatrix}$ to B with position vector $\begin{pmatrix} -12 \\ 15 \\ -4 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} -12 \\ 15 \\ -4 \end{pmatrix} - \begin{pmatrix} 16 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -28 \\ 21 \\ 0 \end{pmatrix}$ **A1**

So an equation for L is $\mathbf{r} = \begin{pmatrix} 16 \\ -6 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -28 \\ 21 \\ 0 \end{pmatrix}$ **M1**

A general point P on the line L has position vector $\begin{pmatrix} 16-28\lambda \\ -6+21\lambda \\ -4 \end{pmatrix}$

So $\overrightarrow{PS} = \begin{pmatrix} 6 \\ 4 \\ -5.2 \end{pmatrix} - \begin{pmatrix} 16-28\lambda \\ -6+21\lambda \\ -4 \end{pmatrix} = \begin{pmatrix} -10+28\lambda \\ 10-21\lambda \\ -1.2 \end{pmatrix}$

If P is the closest point on L to S, then $\overrightarrow{PS}$ must be perpendicular to L, which has direction $\begin{pmatrix} -28 \\ 21 \\ 0 \end{pmatrix}$, so:

$\begin{pmatrix} -10+28\lambda \\ 10-21\lambda \\ -1.2 \end{pmatrix} \cdot \begin{pmatrix} -28 \\ 21 \\ 0 \end{pmatrix} = 0$ **M1**

$\therefore (-10+28\lambda)\times(-28) + (10-21\lambda)\times 21 + (-1.2)\times 0 = 0$ **M1**

$490 - 1225\lambda = 0$

$\lambda = \dfrac{490}{1225}$

$\lambda = \dfrac{2}{5}$ **A1**

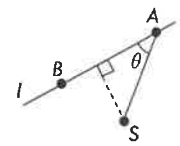
So $\overrightarrow{PS} = \begin{pmatrix} -10+28\lambda \\ 10-21\lambda \\ -1.2 \end{pmatrix} = \begin{pmatrix} -10+28\times\frac{2}{5} \\ 10-21\times\frac{2}{5} \\ -1.2 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.6 \\ -1.2 \end{pmatrix}$ **A1**

Hence $\left|\overrightarrow{PS}\right| = \sqrt{1.2^2 + 1.6^2 + (-1.2)^2} = \sqrt{5.44} = 2.33238\ldots$ km

So the shortest distance from the submarine to the shipwreck is 2.33238… km **A1**

2.33238… > 2 so the submarine does not detect the shipwreck **A1**

Alternatively: If A and B are two points on L, then the shortest distance from S to L is $\left|\overrightarrow{AS}\right| \sin\theta$ where θ is the acute angle between $\overrightarrow{AS}$ and L and is given by $\cos\theta = \left|\dfrac{\overrightarrow{AB}.\overrightarrow{AS}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AS}\right|}\right|$, as shown below:



b) Any suitable criticism **B1**

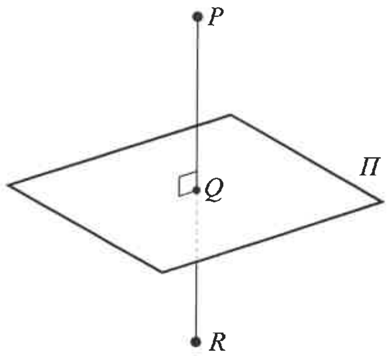
For example:

- The submarine is unlikely to go in a perfectly straight line
- The shipwreck is not really a point but has dimensions
- The submarine is not really a point but has dimensions

[9 Marks]

5. $L: \mathbf{r} = (\mathbf{i}+\mathbf{k}) + \lambda(6\mathbf{i}+\mathbf{j}+\mathbf{k}),\ \Pi : \mathbf{r}.(\mathbf{i}-\mathbf{k}) = -5$

a) Let R be the reflection of $P = (1, 0, 1)$ in the plane Π. Let Q be the midpoint of PR. So P, Q and R are collinear, and the line passing through them is perpendicular to Π (see sketch to the right, for example).



A normal vector to the plane Π is $\mathbf{i} - \mathbf{k}$, so the line through P, Q and R has vector equation:

$$\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\-1\end{pmatrix} \quad \mathbf{M1}$$

Q lies on this line so has position vector $\begin{pmatrix}1+\lambda\\0\\1-\lambda\end{pmatrix}$ for some λ

Q also lies on Π, so:

$$\begin{pmatrix}1+\lambda\\0\\1-\lambda\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix} = -5 \quad \mathbf{M1}$$

$$\therefore\ (1+\lambda)\times 1 + 0\times 0 + (1-\lambda)\times(-1) = -5 \quad \mathbf{M1}$$

$$2\lambda = -5$$

$$\lambda = -\frac{5}{2} \quad \mathbf{A1}$$

Since Q is the midpoint of PR, R corresponds to the point on the line with $\lambda = 2\times\left(-\frac{5}{2}\right) = -5$

So R has position vector $\begin{pmatrix}1\\0\\1\end{pmatrix} + (-5)\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}1-5\\0-0\\1+5\end{pmatrix} = \begin{pmatrix}-4\\0\\6\end{pmatrix}$ **M1**

So R has coordinates $(-4, 0, 6)$ **A1**

b) **Show that the point P lies on the line L, and hence find a vector equation for L', the reflection of the line L in the plane Π**

P has position vector $\begin{pmatrix}1\\0\\1\end{pmatrix}$, which is the point on L corresponding to $\lambda = 0$, so P lies on L **A1**

Let L' be the reflection of L in Π. Then L' passes through both the intersection of L and Π, which we are told is $(-5, -1, 0)$, and the reflection of a point on L.

The point $P = (1, 0, 1)$ lies on L, and its reflection in Π is $(-4, 0, 6)$ by part a)

So L' passes through $(-5, -1, 0)$ and $(-4, 0, 6)$ **M1**

Hence L' has direction $\begin{pmatrix}-4\\0\\6\end{pmatrix} - \begin{pmatrix}-5\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\6\end{pmatrix}$ and passes through $\begin{pmatrix}-5\\-1\\0\end{pmatrix}$, so has vector equation:

$$\mathbf{r} = \begin{pmatrix}-5\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\1\\6\end{pmatrix} \quad \mathbf{A1}$$

Allow any equation of the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$ where $\mathbf{a}$ is the position vector of a point on L' and $\mathbf{b}$ is parallel to $\begin{pmatrix}1\\1\\6\end{pmatrix}$, e.g. $\mathbf{r} = \begin{pmatrix}-4\\0\\6\end{pmatrix} + \mu\begin{pmatrix}-1\\-1\\-6\end{pmatrix}$

[9 Marks]

6. a) The plane has equation $\mathbf{r}.(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=d$, so the vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is perpendicular to the plane, and hence is perpendicular to both $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\5\\0\end{pmatrix}$

So $\begin{pmatrix}a\\b\\c\end{pmatrix}.\begin{pmatrix}1\\3\\-1\end{pmatrix}=a+3b-c=0$

and $\begin{pmatrix}a\\b\\c\end{pmatrix}.\begin{pmatrix}2\\5\\0\end{pmatrix}=2a+5b=0$ **M1**

By letting $a = 1$, we then have to solve the simultaneous equations:

$3b-c=-1$ **(1)**

$5b=-2$ **(2)** **M1**

(2) gives: $b=-\dfrac{2}{5}$

Substituting this into **(1)** gives:

$3\times\left(-\dfrac{2}{5}\right)-c=-1$

$\therefore c=3\times\left(-\dfrac{2}{5}\right)+1=-\dfrac{1}{5}$

So a vector perpendicular to $\begin{pmatrix}1\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\5\\0\end{pmatrix}$ is $\begin{pmatrix}1\\-2/5\\-1/5\end{pmatrix}$ **A1**

Tip: Two equations with three unknowns cannot have a unique solution. Geometrically, they represent the intersection of two planes. So either the planes are parallel and there are no solutions, or they intersect along a line and there are infinitely many solutions. Assuming there is a solution, you can choose a simple value for one of the variables then work out the values of the other variables. This will give you one solution and hence, in this case, a vector that is normal to the plane, but any multiple of this vector will also be perpendicular to the plane, as required.

Multiplying through by 5, this is $\begin{pmatrix}5\\-2\\-1\end{pmatrix}$ **A1**

A known point in the plane has position vector $\begin{pmatrix}3\\1\\1\end{pmatrix}$

So $d=\begin{pmatrix}3\\1\\1\end{pmatrix}.\begin{pmatrix}5\\-2\\-1\end{pmatrix}=3\times5+1\times(-2)+1\times(-1)=12$

So the plane has equation $\mathbf{r}.(5\mathbf{i}-2\mathbf{j}-\mathbf{k})=12$ **A1**

[Allow any integer multiple of this equation]

Tip: Working with column vectors is usually easier and makes mistakes less likely, but be sure to give your final answer in the form requested by the question, in this case using **ijk** notation and with integer coefficients

b) The probe is travelling in a straight line that passes through the points $A = (49, 17, 8)$ and $B = (24, 12, 7)$

So this line has direction $\overrightarrow{AB}=\begin{pmatrix}24\\12\\7\end{pmatrix}-\begin{pmatrix}49\\17\\8\end{pmatrix}=\begin{pmatrix}-25\\-5\\-1\end{pmatrix}$ **A1**

If α is the angle this makes with the normal to the plane, then:

$$\begin{pmatrix}-25\\-5\\-1\end{pmatrix}.\begin{pmatrix}5\\-2\\-1\end{pmatrix}=\left|\begin{pmatrix}-25\\-5\\-1\end{pmatrix}\right|\left|\begin{pmatrix}5\\-2\\-1\end{pmatrix}\right|\cos\alpha$$

$$\begin{pmatrix}-25\\-5\\-1\end{pmatrix}.\begin{pmatrix}5\\-2\\-1\end{pmatrix}=(-25)\times5+(-5)\times(-2)+(-1)\times(-1)=-114 \quad \textbf{M1}$$

$$\left|\begin{pmatrix}-25\\-5\\-1\end{pmatrix}\right|=\sqrt{(-25)^2+(-5)^2+(-1)^2}=\sqrt{651}$$

$$\left|\begin{pmatrix}5\\-2\\-1\end{pmatrix}\right| = \sqrt{5^2+(-2)^2+(-1)^2} = \sqrt{30} \quad \textbf{M1}$$

> **Alternatively:** If a plane has equation **r.n** = k and a line has equation **r** = **a** + λ**b**, then the acute angle between them, θ, satisfies $\sin\theta = \left|\frac{\mathbf{b}.\mathbf{n}}{|\mathbf{b}||\mathbf{n}|}\right|$

So $-114 = \sqrt{651}\sqrt{30}\cos\alpha$ **M1**

Hence $\alpha = \arccos\left(\dfrac{-114}{\sqrt{651}\sqrt{30}}\right) = 144.660...°$

We want the acute angle, so take $\alpha = 180° - 144.660...° = 35.3390...°$ **A1**

This is the angle between the line and the normal to the plane. We want the angle between the line and the plane itself, which is $90° - 35.3390...° = 54.6609...°$ **A1**

$40° < 54.6609...° < 60°$, so the probe will survive entry into Jupiter's atmosphere **A1**

c) Any suitable criticism **B1**

For example:

- The edge of Jupiter's atmosphere is not a flat surface; it is curved
- The probe is unlikely to travel in a straight line when under the influence of Jupiter's gravity

[13 Marks]

TOTAL 60 MARKS

