Complex Numbers II – Test A (50 mins)

Fundamentals ✓**Challenge** Expert

Subtopics: Exponential form of complex numbers, multiplying and dividing complex numbers, de Moivre's theorem, trigonometric identities, sums of series, n^{th} roots of a complex number, solving geometric problems

1. Express each of the following in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give **exact** values for r and θ where possible, or to **3 significant figures** otherwise:
   a) 2i b) −3 + 3i c) 1 − 5i **[5]**

2. Express each of the following in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$:
   a) $e^{\frac{\pi}{7}i}$ b) $4e^{-\frac{6\pi}{5}i}$ c) $\sqrt{2}e^{\frac{13\pi}{3}i}$ **[5]**

3. Express $\dfrac{5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{10\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$, showing your working. **[3]**

4. Use **de Moivre's theorem** to express $\left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^5$ in the form $a + bi$, where $a, b \in \mathbb{R}$ **[3]**

5. Use **de Moivre's theorem** to prove that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$ **[6]**

6. a) Given that $z = \cos\theta + i\sin\theta$, use the result $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ to prove that

   $\cos^3\theta \equiv \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ **[5]**

   b) Hence find the **exact** value of $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta$ **[3]**

7. a) Express $3 + 4i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and where $-\pi < \theta \leq \pi$ is given to **3 decimal places**. **[2]**
   b) Solve the equation $z^3 = 3 + 4i$, expressing the roots in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give your values of r and θ to **3 significant figures**. **[6]**
   c) Illustrate these roots on an Argand diagram. **[1]**

8. A **square** with its centre at the origin is represented in an Argand diagram. One of the vertices is at the point represented by the complex number $\sqrt{7} + 3i$.
   a) Find the coordinates of the other three vertices of the square, giving your answers to **3 significant figures where appropriate**. **[6]**
   b) The complex numbers corresponding to the vertices are squared to form the endpoints of a line. Find the coordinates of the endpoints of this line, and hence find its **length**. **[6]**

9. Given that the **infinite series** A and B defined by $A = \frac{1}{4}\cos\theta + \frac{1}{16}\cos 2\theta + \frac{1}{64}\cos 3\theta + \ldots$ and $B = \frac{1}{4}\sin\theta + \frac{1}{16}\sin 2\theta + \frac{1}{64}\sin 3\theta + \ldots$ are **convergent**, show that $A + iB = \dfrac{e^{i\theta}}{4 - e^{i\theta}}$ **[4]**

10. An **equilateral triangle** with its centre at the origin is represented in an Argand diagram.
    a) One of the vertices is at the point represented by the complex number 6i. Find the coordinates of the other two vertices of the triangle, and illustrate the triangle on an Argand diagram. **[5]**

    The **midpoints** of the sides of this triangle represent the cube roots of the complex number z

    b) Find z **[3]**

TOTAL 63 MARKS

Subtopics: The method of differences, higher derivatives, Maclaurin series, series expansions of compound functions

1. a) Show that $\dfrac{r+1}{r+2}-\dfrac{r}{r+1}\equiv\dfrac{1}{(r+1)(r+2)}$ **[3]**

   b) Hence show that $\sum_{r=1}^{n}\dfrac{1}{(r+1)(r+2)}=\dfrac{n}{2n+4}$, using the **method of differences** **[3]**

2. By using the Maclaurin series of $\cos x$, find the first four terms in the series expansion of $(1+3x)\cos 2x$ **[3]**

3. a) By using a suitable trigonometric identity, or otherwise, find the first **three non-zero** terms in the Maclaurin series expansion of $\sin x\cos x$ **[3]**

   b) State the range of values of x for which the series expansion of $\sin x\cos x$ is valid **[1]**

4. a) Find the first **four non-zero** terms in the Maclaurin series expansions of $\ln(1+3x)$ and $\ln(1-x)$ **[4]**

   b) Hence show that $\ln\left(\dfrac{1+3x}{1-x}\right)=4x-4x^2+ax^3+bx^4+\ldots$, where a and b are constants to be found. State the range of values of x for which this expansion is valid. **[3]**

   c) Hence deduce the series expansion for $\ln\sqrt[4]{\dfrac{1+3x}{1-x}}$, $-\dfrac{1}{3}<x\le\dfrac{1}{3}$, up to and including the term in x^4 **[2]**

   d) Use the first four terms of the series from part b) and a suitable choice of x to find an approximation for $\ln\left(\dfrac{13}{9}\right)$, giving your answer to **3 decimal places.** **[3]**

5. a) Express $\dfrac{2}{(r+1)(r+3)}$ in the form $\dfrac{A}{r+1}+\dfrac{B}{r+3}$, where A and B are constants to be found **[2]**

   b) Hence show that $\sum_{r=1}^{n}\dfrac{2}{(r+1)(r+3)}=\dfrac{5n^2+13n}{6(n+2)(n+3)}$ **[4]**

6. Given that $f(x)=(1+2x)^2\ln(1+2x)$:

   a) show that $f''(x)=a\left(2\ln(1+2x)+3\right)$, where a is an integer to be found **[4]**

   b) find the values of $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ **[5]**

   c) express $f(x)$ in ascending powers of x, **up to and including** the term in x^3 **[2]**

7. a) Use differentiation to show that the first three terms in the Maclaurin series expansion of $e^{-\sin 2x}$ are $1-2x+2x^2$ **[5]**

   b) Hence find $\lim_{x\to 0}\left(\dfrac{e^{-\sin 2x}-1}{x}\right)$ **[1]**

TOTAL 48 MARKS

Further Calculus – Test A (55 mins)

Fundamentals ✓Challenge Expert

Subtopics: Improper integrals, the mean value of a function, differentiating inverse trigonometric functions, integrating with inverse trigonometric functions, integrating using partial fractions, volumes of revolution around the x-axis and the y-axis, volumes of revolution of parametrically defined curves, modelling with volumes of revolution

1. a) Find $\int \frac{1}{4+x^2}\,dx$ **[1]**

   b) Find $\int \frac{1}{4+49x^2}\,dx$, giving your answer in the form $A\arctan(Bx)+c$, where A and B are constants to be found, and c is an arbitrary constant **[3]**

2. By considering their behaviour at each limit, show that the following improper integrals **diverge**:

   a) $\int_4^{\infty} \frac{1}{\sqrt{x}}\,dx$ b) $\int_0^{\pi/2} \tan x\,dx$ **[6]**

3. Use the substitution $x = \frac{1}{5}\sin\theta$ to find $\int \frac{1}{\sqrt{1-25x^2}}\,dx$ in the form $A\arcsin(Bx)+c$, where A and B are constants to be found, and c is an arbitrary constant. **[4]**

4. Use **implicit differentiation** to show that $\frac{d}{dx}\arccos\left(\frac{1}{x}\right) = \frac{1}{x\sqrt{x^2-1}}$ **[4]**

5. Evaluate the improper integral $\int_0^{\infty} 3x^2 e^{-x^3}\,dx$ **[3]**

6. a) Find $\int_0^{\pi/2} \frac{\cos x}{(\sin x+1)^2}\,dx$ **[3]**

   b) Find the **exact mean value** of $f(x) = \frac{\cos x}{(\sin x+1)^2}$ over the interval $\left[0, \frac{\pi}{2}\right]$ **[2]**

   c) Hence state the **exact mean value** of $f(x)+3$ over the interval $\left[0, \frac{\pi}{2}\right]$ **[1]**

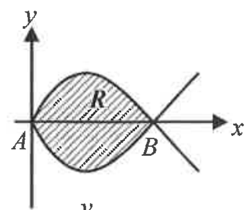
7. **Using partial fractions**, show that $\int \frac{20x-15}{(x^2+9)(x+6)}\,dx = P\ln\left|\frac{\sqrt{x^2+9}}{x+6}\right| + Q\arctan\left(\frac{x}{3}\right)+c$,

   where P and Q are constants to be found, and c is an arbitrary constant. **[5]**

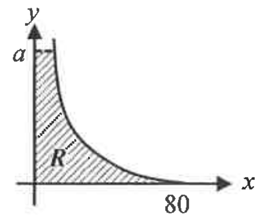
8. The diagram to the right shows the curve with equation $2y^2 = x\cos\left(\frac{x}{2}\right)+x,\ 0 \le x \le 3\pi$

   a) Find the x-coordinates of A and B **[2]**

   b) Find the **exact volume** of the solid formed when the shaded region R is rotated 2π radians about the x-axis **[5]**



9. The diagram to the right shows the curve C with parametric equations $x = \frac{400}{11+t},\ y = \frac{1}{2}t+3,\ t \ge -6$. A table base is modelled as the solid of revolution formed when the finite region R bounded by C, the x-axis, the y-axis and the line $y=a$ is rotated 2π radians about the y-axis. Given that the units of x and y are centimetres, and that the table base has volume $15\,500\pi$ cm^3, find the **exact** value of a. **[8]**



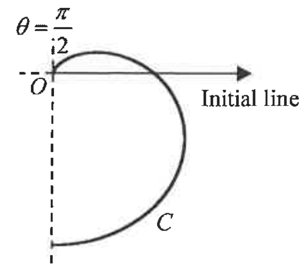
TOTAL 47 MARKS

Subtopics: Polar coordinates and equations, sketching curves, area enclosed by a polar curve, tangents to polar curves

1. Find **Cartesian** equations for the following curves:
   a) $r = 3$ **[2]**
   b) $r\sin\theta = 1$ **[2]**
   c) $r - 2\sec\theta = 0$ **[2]**
   d) $r\tan\theta - r = 0$ **[2]**

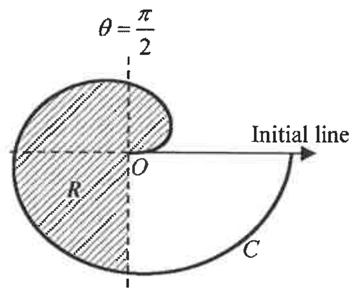
2. On separate diagrams, sketch the following curves for $0 \le \theta < 2\pi$:
   a) $\theta = \dfrac{2\pi}{3}$ **[2]**
   b) $r = \dfrac{\theta}{2\pi}$ **[2]**
   c) $r = 1 - \cos\theta$ **[2]**
   d) $r^2 = \sin 2\theta$ **[2]**

3. The curve C sketched to the right has equation $r = 1 - \sin\theta$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$



   a) Find the exact polar coordinates of the point on C where the tangent is **parallel** to the initial line. **[4]**
   b) Find the exact polar coordinates of the point on C where the tangent is **perpendicular** to the initial line. **[4]**

4. C is the curve given by the equation $r = \sqrt{\theta}$ for $0 \le \theta \le 2\pi$, as illustrated to the right. Find the area of the shaded region R. Give your answer in the form $a\pi^n$, where a is a **rational** number and n is an **integer**. **[4]**



5. Find **polar** equations in the stated forms for the following curves:
   a) $x^2 + 2y^2 = 3$ in the form $r^2 = \mathrm{f}(\theta)$ **[2]**
   b) $y = 2x + 1$ in the form $r = \mathrm{f}(\theta)$ **[3]**
   c) $x^2 - y^2 = 1$ in the form $r^2 = \mathrm{f}(\theta)$, where f is a **single trigonometric function** **[3]**

6. The constant $a > 1$ is such that the area enclosed by the curve $r = a - \sin\theta$ is $\dfrac{33}{2}\pi$. Find a. **[5]**

7. Let C be the curve $r = 1 + \cos\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$. Find the **exact Cartesian equation** of the tangent to C that is **parallel** to the initial line. **[5]**

8. a) On the **same diagram**, sketch the curves with polar equations $r = 3 + 2\cos\theta$ and $r = 4$ for $0 \le \theta < 2\pi$ **[4]**
   b) Find the exact polar coordinates of the points where these two curves intersect **[4]**

TOTAL 54 MARKS

Hyperbolic Functions – Test A (50 mins)

Fundamentals ✓**Challenge** Expert

Subtopics: Hyperbolic functions, inverse hyperbolic functions, identities and equations, differentiating hyperbolic functions, integrating hyperbolic functions

1. **Use the exponential definition** of $\cosh x$ to find the **exact** value of $\cosh(\ln 2)$, showing your working. **[3]**

2. a) On the **same diagram**, sketch the graphs of $y = \tanh x$ and $y = \tanh 2x$, indicating any asymptotes **[3]**
   b) State the **range** of $f(x) = \tanh 2x$, where $x \in \mathbb{R}$ **[1]**

3. Find: a) $\int \frac{2}{\sqrt{x^2 - 1}}\,dx$ b) $\int \frac{5}{\sqrt{4 + x^2}}\,dx$ **[2]**

4. a) Find $\frac{d}{dx}\sinh 2x$ **[1]**
   b) Find $\frac{d^2}{dx^2}\sinh 2x \cosh x$ **[5]**

5. Using the result $\cosh^2 x - \sinh^2 x \equiv 1$, **show that** $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$ for $x \neq 0$ **[3]**

6. Find: a) $\int \sinh 4x\,dx$ b) $\int 6e^{-x}\sinh 4x\,dx$ **[5]**
   Hint for part b): Write sinh 4x in exponential form

7. Solve the equation $6\cosh x - 2\sinh x = 9$, giving your answers as **natural logarithms**. **[6]**

8. a) By writing $\operatorname{arsinh} x = y$, show that $y = \ln\left(x + \sqrt{x^2 + 1}\right)$ **[5]**
   b) Hence express $\operatorname{arsinh} 2$ as a **natural logarithm** **[2]**

9. Sketch the graph of $y = \operatorname{arcosh}(x - 1)$, indicating where the graph meets the x-axis. **[2]**

10. **Prove** that $\cosh^2 x + \sinh^2 x \equiv \cosh 2x$ using the definitions of $\sinh x$ and $\cosh x$ **[3]**

11. a) Given that $\cosh x = \frac{3}{2}$, find the **two possible exact** values of $\tanh x$ **[3]**
    b) **Use the exponential definition** of $\cosh x$ to find, correct to **2 decimal places**, the values of x for which $\cosh x = \frac{3}{2}$. Show your working. **[5]**

12. Find the **exact coordinates** of the stationary point on the curve with equation $y = 12\cosh x - 6\sinh x$ **[6]**

13. By rewriting the denominator in the form $\sqrt{a\left((x+b)^2 - c\right)}$, find $\int \frac{1}{\sqrt{4x^2 - 16x}}\,dx$ **[5]**

14. Use the substitution $x = \frac{1}{2}(1 + 5\cosh u)$ to find $\int \frac{1}{\sqrt{x^2 - x - 6}}\,dx$ **[6]**

TOTAL 66 MARKS

Subtopics: First- and second-order differential equations, boundary conditions, modelling, simple harmonic motion, damped and forced harmonic motion, coupled first-order simultaneous differential equations

1. Use the integrating factor x^3 to find the **general solution** of the differential equation $\frac{dy}{dx}+\frac{3y}{x}=4-\frac{6}{x}$.
Give your answer in the form $y = f(x)$. **[4]**

2. a) Find the **general solutions** to the following differential equations, where y is a function of x:
   i) $y'' + 9y' + 14y = 0$ **[3]**
   ii) $y'' - 6y' + 9y = 0$ **[3]**
   b) Find the **particular solution** to the differential equation $y'' + 4y = 0$ that satisfies $y = 5$ when $x = 0$ and $y = -6$ when $x = \frac{\pi}{4}$ **[5]**

3. A particle is moving along a straight line. Its displacement in metres from a fixed point O at time t seconds is denoted by x, and satisfies the differential equation $\ddot{x} = -16x$
   a) Describe the motion of the particle **[1]**
   b) Show that the **general solution** to the differential equation is $x = A\cos 4t + B\sin 4t$ **[3]**
   When $t = 0$, the particle has displacement –5 m and velocity 12 m s^{-1}
   c) Find the **particular solution** for x in terms of t **[3]**

4. Consider the differential equation $y'' - 4y' + 3y = 6x + 1$
   a) Find the **complementary function** for this equation. **[3]**
   b) Find a **particular integral** for this equation. **[4]**
   c) Hence write down the **general solution** for the equation. **[1]**

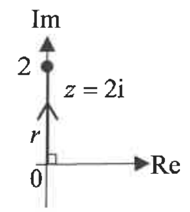
5. A survey counts the number of mice, x, and the number of owls, y, in a certain region.
After t years, the values of x and y are modelled by the coupled differential equations:
$$\frac{dx}{dt} = x - \frac{5}{3}y$$
$$\frac{dy}{dt} = \frac{2}{3}x - y$$
   a) Show that $\frac{d^2x}{dt^2} + \frac{x}{9} = 0$ **[4]**
   b) Hence find a **general solution** for the number of mice in terms of t **[3]**

6. a) Find the **general solution** to the differential equation $\frac{dy}{dx} - \frac{y}{x} = x$ **[6]**
   b) On the **same axes**, sketch three different members of the family of solution curves to this differential equation **[2]**

7. Bethany is bungee jumping over a river. Her distance in metres above some fixed point O at a time t seconds after jumping is denoted by x. Bethany models x by the equation $\ddot{x} + 6\dot{x} + 9x = 0$. When $t = 0$, Bethany is 15 m above O and has a velocity of 5 m s^{-1} upwards.
   a) Find an expression for x in terms of t **[6]**
   b) Sketch the curve of x against t for $t \geq 0$ **[2]**
   c) Give one criticism of Bethany's model **[1]**

TOTAL 54 MARKS

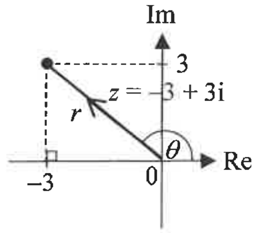
1. a)



$r = 2,\ \theta = \arg z = \dfrac{\pi}{2}$

$\therefore z = 2e^{\frac{\pi}{2}i}$ **A1**

b)



Technique: You can write a complex number z in the form $z = re^{i\theta}$ using the facts that $r = |z|$ and $\theta = \arg z$. Sketching an Argand diagram will help you to find r and θ.

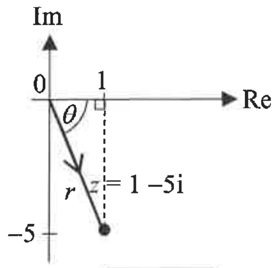
$r = |z| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

$\theta = \arg z = \pi - \arctan\left(\dfrac{3}{3}\right) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$ **M1**

$\therefore z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$ (also accept $\sqrt{18}$ for $3\sqrt{2}$) **A1**

Tip: Learn how to use the Complex menu on your calculator to convert between the different forms of complex numbers

c)



$r = |z| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

$\theta = \arg z = -\arctan\left(\dfrac{5}{1}\right) = -1.37340... = -1.37$ (3 s.f.) **M1**

$\therefore z = \sqrt{26}e^{-1.37i}$ **A1** **[5 Marks]**

2. a) $e^{\frac{\pi}{7}i} = \cos\dfrac{\pi}{7} + i\sin\dfrac{\pi}{7}$ $\left(\text{also allow } 1\left(\cos\dfrac{\pi}{7} + i\sin\dfrac{\pi}{7}\right)\right)$ **A1**

Technique: Use the facts that $\cos\theta = \cos(\theta + 2\pi)$ and $\sin\theta = \sin(\theta + 2\pi)$ to find a value of θ in the interval $-\pi < \theta \leq \pi$

b) $-\dfrac{6\pi}{5} + 2\pi = \dfrac{4\pi}{5}$ **M1**

So $4e^{-\frac{6\pi}{5}i} = 4e^{\frac{4\pi}{5}i} = 4\left(\cos\dfrac{4\pi}{5} + i\sin\dfrac{4\pi}{5}\right)$ **A1**

Hint: You may have to add or subtract a multiple of 2π in order to find a value of θ in the given interval

c) $\dfrac{13\pi}{3} - 2\pi = \dfrac{7\pi}{3}$

$\dfrac{7\pi}{3} - 2\pi = \dfrac{\pi}{3}$ **M1**

So $\sqrt{2}e^{\frac{13\pi}{3}i} = \sqrt{2}e^{\frac{\pi}{3}i} = \sqrt{2}\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$ **A1** **[5 Marks]**

3. $\dfrac{5\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)}{10\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)}$ is of the form $\frac{z_1}{z_2}$, with $z_1=5\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)$ and $z_2=10\left(\cos\frac{2\pi}{3}+\mathrm{i}\sin\frac{2\pi}{3}\right)$

So $\frac{z_1}{z_2}=\dfrac{5\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{10\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}}$ **M1**

> **Technique:** First write both numbers in the form $r\mathrm{e}^{\mathrm{i}\theta}$. Then divide the moduli and subtract the arguments to find the result of the division.

$=\frac{5}{10}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}-\frac{2\pi}{3}\right)}$ **M1**

$=\frac{1}{2}\mathrm{e}^{-\frac{5\pi}{12}\mathrm{i}}$ **A1** **[3 Marks]**

4. $\left(2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)\right)^5=2^5\left(\cos\left(5\times\frac{\pi}{6}\right)+\mathrm{i}\sin\left(5\times\frac{\pi}{6}\right)\right)$ **M1**

> **Hint:** de Moivre's theorem states that $(r(\cos\theta+\mathrm{i}\sin\theta))^n = r^n(\cos n\theta+\mathrm{i}\sin n\theta)$

$=32\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$ **M1**

$=-16\sqrt{3}+16\mathrm{i}$ **A1** **[3 Marks]**

5. **Prove that $\sin 3\theta\equiv 3\sin\theta-4\sin^3\theta$**

$(\cos\theta+\mathrm{i}\sin\theta)^3=\cos 3\theta+\mathrm{i}\sin 3\theta$ by de Moivre's theorem **M1**

> **Technique:** Expand the left-hand side using the binomial expansion and simplify the powers of i

$(\cos\theta+\mathrm{i}\sin\theta)^3=\cos^3\theta+{}^3C_1\cos^2\theta\times\mathrm{i}\sin\theta+{}^3C_2\cos\theta\times(\mathrm{i}\sin\theta)^2+(\mathrm{i}\sin\theta)^3$ **M1**

$=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta+3\mathrm{i}^2\cos\theta\sin^2\theta+\mathrm{i}^3\sin^3\theta$

$=\cos^3\theta+3\mathrm{i}\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-\mathrm{i}\sin^3\theta$ **A1**

Equating imaginary parts:

$\sin 3\theta\equiv 3\cos^2\theta\sin\theta-\sin^3\theta$

$\equiv\sin\theta(3\cos^2\theta-\sin^2\theta)$ **M1**

$\equiv\sin\theta(3(1-\sin^2\theta)-\sin^2\theta)$ **M1**

> **Technique:** Use the identity $\cos^2\theta+\sin^2\theta\equiv 1$

$\equiv\sin\theta(3-3\sin^2\theta-\sin^2\theta)$

$\equiv\sin\theta(3-4\sin^2\theta)$

$\equiv 3\sin\theta-4\sin^3\theta$ **A1** **[6 Marks]**

6. a) **Given that $z=\cos\theta+\mathrm{i}\sin\theta$, use the result $z^n+\frac{1}{z^n}=2\cos n\theta$ to prove that $\cos^3\theta\equiv\frac{1}{4}(\cos 3\theta+3\cos\theta)$**

For $n=1$, $z+\frac{1}{z}=2\cos\theta$

So $\left(z+\frac{1}{z}\right)^3=(2\cos\theta)^3=8\cos^3\theta$ **M1**

Also $\left(z+\frac{1}{z}\right)^3=z^3+{}^3C_1z^2\left(\frac{1}{z}\right)+{}^3C_2z\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3$ **M1**

> **Technique:** Expand the left-hand side using the binomial expansion and simplify the powers of z

$=z^3+3z^2\left(\frac{1}{z}\right)+3z\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^3}\right)$

$=z^3+3z+\frac{3}{z}+\frac{1}{z^3}$

$=\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)$ **M1**

> **Technique:** Group terms to the same power of z and rewrite the expression in terms of $\cos n\theta$

$=2\cos 3\theta+3(2\cos\theta)=2\cos 3\theta+6\cos\theta$ by using $n=3$ and $n=1$ **M1**

$\therefore 8\cos^3\theta\equiv 2\cos 3\theta+6\cos\theta$

$\therefore \cos^3\theta\equiv\frac{2}{8}\cos 3\theta+\frac{6}{8}\cos\theta\equiv\frac{1}{4}(\cos 3\theta+3\cos\theta)$ **A1**

b) $$\int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{4}(\cos 3\theta + 3\cos\theta)\, d\theta$$

$$= \left[\frac{1}{12}\sin 3\theta + \frac{3}{4}\sin\theta\right]_0^{\frac{\pi}{2}} \quad \textbf{M1}$$

$$= \frac{1}{12}\sin\frac{3\pi}{2} + \frac{3}{4}\sin\frac{\pi}{2} - 0 \quad \textbf{M1}$$

$$= -\frac{1}{12} + \frac{3}{4} = \frac{2}{3} \quad \textbf{A1}$$

[8 Marks]

7. a) Let $w = 3 + 4i$, so $|w| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, $\arg w = \arctan\left(\frac{4}{3}\right) = 0.927295... = 0.927$ (3 s.f.) **M1**

$\therefore 3 + 4i = 5(\cos 0.927 + i\sin 0.927)$ **A1**

b) Let $z^3 = 5(\cos(0.927 + 2k\pi) + i\sin(0.927 + 2k\pi))$ **M1**

> **Technique:** Use the facts that $\cos\theta = \cos(\theta + 2k\pi)$ and $\sin\theta = \sin(\theta + 2k\pi)$ for any integer k

Also $z^3 = (r(\cos\theta + i\sin\theta))^3 = r^3(\cos 3\theta + i\sin 3\theta)$ by de Moivre's theorem **M1**

So $5 = r^3 \therefore r = 1.70997... = 1.71$ (3 s.f.) and $0.927295... + 2k\pi = 3\theta$ **M1**

For $k = 0$, $\theta = \frac{0.927295...}{3} = 0.309098... = 0.309$ (3 s.f.)

> **Technique:** Obtain possible values for θ by considering different values of k

So $z_1 = 1.71(\cos 0.309 + i\sin 0.309)$ **A1**

For $k = 1$, $\theta = \frac{7.21048...}{3} = 2.40349... = 2.40$ (3 s.f.)

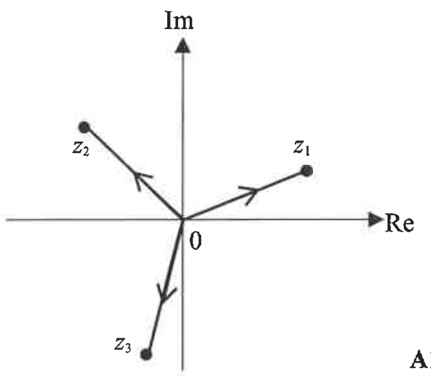
So $z_2 = 1.71(\cos 2.40 + i\sin 2.40)$ **A1**

For $k = 2$, $\theta = \frac{13.4936...}{3} = 4.49788...$, but this is not in the interval $-\pi < \theta \le \pi$, so use

$\theta = 4.49788... - 2\pi = -1.78529... = -1.79$ (3 s.f.)

So $z_3 = 1.71(\cos(-1.79) + i\sin(-1.79))$ **A1**

c)



A1 **[9 Marks]**

8. a) Let $z = \sqrt{7} + 3i$, so $|z| = \sqrt{(\sqrt{7})^2 + 3^2} = \sqrt{16} = 4$, $\arg z = \arctan\left(\frac{3}{\sqrt{7}}\right) = 0.848062...$ **M1**

$\therefore z = 4e^{0.848062...i}$ **M1**

The fourth roots of unity are 1, ω, ω^2 and ω^3, where $\omega = e^{\frac{2\pi}{4}i} = e^{\frac{\pi}{2}i}$ **M1**

> **Technique:** Because the vertices of a square with its centre at the origin represent the fourth roots of a complex number, you can use the fourth roots of unity and a given vertex to work out all the others

The given vertex represents $4e^{0.848062...i}$, so the other vertices represent:

$z\omega = 4e^{i\left(0.848062...+\frac{\pi}{2}\right)} = 4e^{2.41885...i}$, i.e. $(-3,\ 2.64575...) = (-3,\ 2.65)$ (3 s.f.) **A1**

$z\omega^2 = 4e^{i(0.848062...+\pi)} = 4e^{3.98965...i}$, i.e. $(-2.64575...,\ -3) = (-2.65,\ -3)$ (3 s.f.) **A1**

$z\omega^3 = 4e^{i\left(0.848062...+\frac{3\pi}{2}\right)} = 4e^{5.56045...i}$, i.e. $(3,\ -2.64575...) = (3,\ -2.65)$ (3 s.f.) **A1**

> **Alternatively:** The fourth roots of unity are 1, i, −1 and −i, so in this case you could just multiply $z = \sqrt{7} + 3i$ by each of these

b) $\left(\sqrt{7}+3\mathrm{i}\right)^2=\left(-2.64575...-3\mathrm{i}\right)^2=7+15.8745...\mathrm{i}+9\mathrm{i}^2=-2+15.8745...\mathrm{i}$ **M1**

So one endpoint is at $(-2,\ 15.8745...)$ **A1**

$\left(-3+2.64575...\mathrm{i}\right)^2=\left(3-2.64575...\mathrm{i}\right)^2=9-15.8745...\mathrm{i}+7\mathrm{i}^2=2-15.8745...\mathrm{i}$ **M1**

So the other endpoint is at $(2,\ -15.8745...)$ **A1**

Length of the line is $\sqrt{(2--2)^2+(-15.8745...-15.8745...)^2}$ **M1**

$=\sqrt{4^2+(-31.7490...)^2}=\sqrt{1024}=32$ **A1** **[12 Marks]**

9. **Show that $A+\mathrm{i}B=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{4-\mathrm{e}^{\mathrm{i}\theta}}$**

$A=\frac{1}{4}\cos\theta+\frac{1}{16}\cos 2\theta+\frac{1}{64}\cos 3\theta+...$ and $B=\frac{1}{4}\sin\theta+\frac{1}{16}\sin 2\theta+\frac{1}{64}\sin 3\theta+...$

$$A+\mathrm{i}B=\left(\frac{1}{4}\cos\theta+\frac{1}{16}\cos 2\theta+\frac{1}{64}\cos 3\theta+...\right)+\mathrm{i}\left(\frac{1}{4}\sin\theta+\frac{1}{16}\sin 2\theta+\frac{1}{64}\sin 3\theta+...\right)$$

$$=\frac{1}{4}(\cos\theta+\mathrm{i}\sin\theta)+\frac{1}{4^2}(\cos 2\theta+\mathrm{i}\sin 2\theta)+\frac{1}{4^3}(\cos 3\theta+\mathrm{i}\sin 3\theta)+...\quad \textbf{M1}$$

$$=\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}+\frac{1}{4^2}\mathrm{e}^{2\mathrm{i}\theta}+\frac{1}{4^3}\mathrm{e}^{3\mathrm{i}\theta}+...\quad \textbf{M1}$$

This is an infinite geometric series with $a=\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}$ and $r=\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}$

So $S_\infty=\dfrac{\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}}{1-\frac{1}{4}\mathrm{e}^{\mathrm{i}\theta}}$ **M1**

> **Technique:** Remember that the sum of a convergent infinite geometric series with first term a and common ratio r is $S_\infty=\dfrac{a}{1-r}$

$=\dfrac{\mathrm{e}^{\mathrm{i}\theta}}{4-\mathrm{e}^{\mathrm{i}\theta}}$ **A1** **[4 Marks]**

10. a) Let the vertex A represent the complex number $6\mathrm{i}=6\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$ **M1**

The cube roots of unity are 1, ω and ω^2, where $\omega=\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$

> **Technique:** Because the vertices of an equilateral triangle with its centre at the origin represent the cube roots of a complex number, you can use the cube roots of unity and a given vertex to find the other two

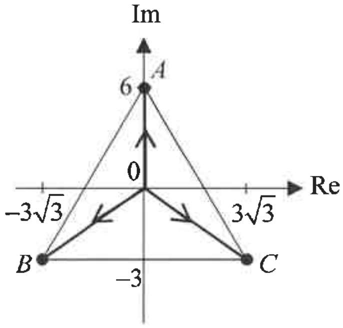
So the other vertices of the triangle (B and C) are given by

B: $6\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\times\omega=6\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\times\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}=6\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)}=6\mathrm{e}^{\frac{7\pi}{6}\mathrm{i}}\left(=6\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}\right)$ **M1**

i.e. $\left(-3\sqrt{3},\ -3\right)$ **A1**

and C: $6\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\times\omega^2=6\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\times\mathrm{e}^{\frac{4\pi}{3}\mathrm{i}}=6\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}+\frac{4\pi}{3}\right)}=6\mathrm{e}^{\frac{11\pi}{6}\mathrm{i}}\left(=6\mathrm{e}^{-\frac{\pi}{6}\mathrm{i}}\right)$, i.e. $\left(3\sqrt{3},\ -3\right)$ **A1**

> **Alternatively:** Once you have worked out the coordinates of either B or C, just use the symmetry properties of the equilateral triangle to work out the coordinates of the other



A1

b) The midpoint of BC is $(0,\ -3)$ **M1**

> **Hint:** The midpoint of BC is the easiest of the three midpoints to find since BC is horizontal, but you could also find the midpoint of AB or AC, and then convert it to a complex number and cube it to find z

So one of the cube roots of z is $-3\mathrm{i}$ **M1**

$\therefore z=(-3\mathrm{i})^3=-27\mathrm{i}^3=27\mathrm{i}$ **A1** **[8 Marks]**

TOTAL 63 MARKS

1. a) **Show that** $\frac{r+1}{r+2}-\frac{r}{r+1}\equiv\frac{1}{(r+1)(r+2)}$

$$\frac{r+1}{r+2}-\frac{r}{r+1}=\frac{(r+1)^2-r(r+2)}{(r+1)(r+2)} \quad \textbf{M1}$$

Technique: Write the left-hand side as a single fraction by finding a common denominator

$$=\frac{r^2+2r+1-r^2-2r}{(r+1)(r+2)} \quad \textbf{M1}$$

$$=\frac{1}{(r+1)(r+2)} \quad \textbf{A1}$$

b) **Show that** $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\frac{n}{2n+4}$**, using the method of differences**

$$\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{r+1}{r+2}-\frac{r}{r+1}\right)$$

Hint: Write out enough unsimplified terms of the sum to see which ones cancel and which ones remain

Let $r=1$: $\frac{2}{3}-\frac{1}{2}$

$r=2$: $\frac{3}{4}-\frac{2}{3}$

$r=3$: $\frac{4}{5}-\frac{3}{4}$

$\vdots$

$r=n$: $\frac{n+1}{n+2}-\frac{n}{n+1}$

So the sum is $\frac{n+1}{n+2}-\frac{1}{2}$ **M1**

$$\text{So } \sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}=\frac{n+1}{n+2}-\frac{1}{2}$$

$$=\frac{2(n+1)-(n+2)}{2(n+2)} \quad \textbf{M1}$$

$$=\frac{2n+2-n-2}{2n+4}=\frac{n}{2n+4} \quad \textbf{A1}$$

[6 Marks]

2. $$\cos 2x=1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\ldots \quad \textbf{M1}$$

$$=1-2x^2+\frac{2}{3}x^4-\ldots$$

Hint: You only need to expand enough to be able to select the first four non-zero terms, so in this case even the x^4 term is not necessary

$$\therefore (1+3x)\cos 2x=(1+3x)\left(1-2x^2+\frac{2}{3}x^4-\ldots\right) \quad \textbf{M1}$$

$$=1-2x^2+\frac{2}{3}x^4+3x-6x^3+2x^5+\ldots$$

The first four terms are $1+3x-2x^2-6x^3$ **A1** **[3 Marks]**

Technique: Use the identity $\sin 2x\equiv 2\sin x\cos x$

3. a) $\sin x\cos x\equiv\frac{1}{2}\sin 2x$ **M1**

Alternatively: Find the series expansion as the product of the expansions of sin x and cos x, using the first three terms of each expansion, and ignoring any terms involving higher powers of x

$$\frac{1}{2}\sin 2x=\frac{1}{2}\left(2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\ldots\right) \quad \textbf{M1}$$

$$=\frac{1}{2}\left(2x-\frac{4}{3}x^3+\frac{4}{15}x^5-\ldots\right)=x-\frac{2}{3}x^3+\frac{2}{15}x^5-\ldots \quad \textbf{A1}$$

b) The expansion of $\sin x\cos x$ is valid for all values of x **A1** **[4 Marks]**

4. a) $\ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \dots$ **M1**

$= 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots$ **A1**

$\ln(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \dots$ **M1**

$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots$ **A1**

> **Technique:** Replace x with $3x$ in the series expansion for $\ln(1+x)$, and simplify. Similarly, replace x with $-x$ for the second expansion.

> **Technique:** Use the division law of logarithms, $\log\frac{a}{b} = \log a - \log b$

b) $\ln\left(\frac{1+3x}{1-x}\right) = \ln(1+3x) - \ln(1-x) = \left(3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \dots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)$ **M1**

$= 4x - 4x^2 + \frac{28}{3}x^3 - 20x^4 + \dots \left(\text{i.e. } a = \frac{28}{3},\ b = -20\right)$ **A1**

The expansion of $\ln\left(\frac{1+3x}{1-x}\right)$ is valid provided both $-1 < 3x \le 1$, i.e. $-\frac{1}{3} < x \le \frac{1}{3}$,

and $-1 < -x \le 1$, i.e. $-1 \le x < 1$, so the expansion is only valid for $-\frac{1}{3} < x \le \frac{1}{3}$ **A1**

> **Technique:** Take the stricter of the two conditions as the final condition for validity

c) $\ln\sqrt[4]{\frac{1+3x}{1-x}} = \ln\left(\left(\frac{1+3x}{1-x}\right)^{\frac{1}{4}}\right) = \frac{1}{4}\ln\left(\frac{1+3x}{1-x}\right)$ **M1**

$= \frac{1}{4}\left(4x - 4x^2 + \frac{28}{3}x^3 - 20x^4 + \dots\right) = x - x^2 + \frac{7}{3}x^3 - 5x^4 + \dots$ **A1**

> **Technique:** Use the power law of logarithms, $\log a^b = b\log a$

d) We want to approximate $\ln\left(\frac{13}{9}\right)$, so let $\frac{1+3x}{1-x} = \frac{13}{9}$

$9(1+3x) = 13(1-x)$

$9 + 27x = 13 - 13x$

$40x = 4 \therefore x = \frac{1}{10}$ **A1**

> **Technique:** Find the value of x which makes the expression equal to the one we are trying to approximate, and then use that value of x in the first four terms of the series expansion, making sure this value of x is within the range of validity for the expansion

So $\ln\left(\frac{13}{9}\right) = \ln\left(\frac{1+3\times\frac{1}{10}}{1-\frac{1}{10}}\right) \approx 4\times\frac{1}{10} - 4\left(\frac{1}{10}\right)^2 + \frac{28}{3}\left(\frac{1}{10}\right)^3 - 20\left(\frac{1}{10}\right)^4$ **M1**

$= \frac{551}{1500} = 0.367333\dots = 0.367$ (3 d.p.) **A1** **[12 Marks]**

5. a) Let $\frac{2}{(r+1)(r+3)} \equiv \frac{A}{r+1} + \frac{B}{r+3} \equiv \frac{A(r+3)+B(r+1)}{(r+1)(r+3)}$

So $2 \equiv A(r+3) + B(r+1)$

Substituting $r = -3$, $2 = -2B \therefore B = -1$

Substituting $r = -1$, $2 = 2A \therefore A = 1$ **M1**

So $\frac{2}{(r+1)(r+3)} \equiv \frac{1}{r+1} - \frac{1}{r+3}$ **A1**

> **Alternatively:** You could also equate coefficients of r and equate constants on both sides of the equation to find the values of A and B

b) **Show that** $\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{5n^2+13n}{6(n+2)(n+3)}$

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \sum_{r=1}^{n}\left(\frac{1}{r+1} - \frac{1}{r+3}\right)$$

Let $r=1: \frac{1}{2} - \cancel{\frac{1}{4}}$

Hint: As the sum involves $r + 1$ and $r + 3$, you will need to include more terms in order to see the pattern. It is also useful to include the last few terms at the end of the series.

$r=2: \frac{1}{3} - \cancel{\frac{1}{5}}$

$r=3: \cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}$

$r=4: \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}$

$\vdots$

$r=n-2: \cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n+1}}$

$r=n-1: \cancel{\frac{1}{n}} - \frac{1}{n+2}$

$r=n: \cancel{\frac{1}{n+1}} - \frac{1}{n+3}$

So the sum is $\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}$ **M1A1**

$$=\frac{5}{6}-\frac{1}{n+2}-\frac{1}{n+3}$$

Technique: Write as a single fraction by finding a common denominator and then simplifying

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{5(n+2)(n+3)-6(n+3)-6(n+2)}{6\times(n+2)\times(n+3)} \quad \textbf{M1}$$

$$=\frac{5(n^2+5n+6)-6n-18-6n-12}{6(n+2)(n+3)} = \frac{5n^2+25n+30-12n-30}{6(n+2)(n+3)}$$

$$=\frac{5n^2+13n}{6(n+2)(n+3)} \quad \textbf{A1}$$ **[6 Marks]**

6. a) $f(x) = (1+2x)^2 \ln(1+2x)$

$f'(x) = 2\times2(1+2x)\ln(1+2x) + \frac{(1+2x)^2\times2}{1+2x}$ **M1**

Technique: Use the product rule and the chain rule to differentiate f(x) twice

$= 4(1+2x)\ln(1+2x) + 2(1+2x)$

$= 4(1+2x)\ln(1+2x) + 2 + 4x$ **A1**

$f''(x) = 4\times2\ln(1+2x) + \frac{4(1+2x)\times2}{1+2x} + 4$ **M1**

$= 8\ln(1+2x) + 8 + 4$

$= 4(2\ln(1+2x)+3)$ (i.e. $a=4$) **A1**

b) $f(0) = 1^2 \times \ln 1 = 0$ **A1**

$f'(0) = 4\times1\times\ln1 + 2 + 0 = 2$ **A1**

$f''(0) = 4(2\ln 1 + 3) = 12$ **A1**

$f'''(x) = \frac{4\times2\times2}{1+2x} = \frac{16}{1+2x}$ **M1**

$\therefore f'''(0) = \frac{16}{1} = 16$ **A1**

c) $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$

So $f(x) = 0 + x \times 2 + \frac{x^2}{2!} \times 12 + \frac{x^3}{3!} \times 16 + \ldots$ **M1**

$= 2x + 6x^2 + \frac{8}{3}x^3 + \ldots$ **A1** **[11 Marks]**

7\. a) **Use differentiation to show that the first three terms in the series expansion of $e^{-\sin 2x}$ are $1 - 2x + 2x^2$**

$f(x) = e^{-\sin 2x} \therefore f(0) = e^{-\sin 0} = 1$ **M1**

$f'(x) = -2\cos 2x \times e^{-\sin 2x} \therefore f'(0) = -2\cos 0 \times e^{-\sin 0} = -2$ **M1**

Technique: Use the chain rule and the product rule to differentiate $f(x)$ twice

$f''(x) = 4\sin 2x \times e^{-\sin 2x} - 2\cos 2x \times (-2\cos 2x) \times e^{-\sin 2x} = 4e^{-\sin 2x}(\sin 2x + \cos^2 2x)$ **M1**

$\therefore f''(0) = 4e^{-\sin 0}(\sin 0 + \cos^2 0) = 4 \times 1 = 4$ **M1**

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

So $e^{-\sin 2x} = 1 - 2x + \frac{x^2}{2!} \times 4 + \ldots = 1 - 2x + 2x^2 + \ldots$ **A1**

b) $\frac{e^{-\sin 2x} - 1}{x} = \frac{1 - 2x + 2x^2 + \ldots - 1}{x} = -2 + 2x + \ldots$

So $\lim_{x \to 0}\left(\frac{e^{-\sin 2x} - 1}{x}\right) = -2$ **A1** **[6 Marks]**

TOTAL 48 MARKS

1. a) $\int \frac{1}{4+x^2}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)+c$ **A1**

b) $\int \frac{1}{4+49x^2}\,dx = \int \frac{1}{49\left(\frac{4}{49}+x^2\right)}\,dx$ **M1** ◄

Technique: Write $4 + 49x^2$ in the form $k(a^2 + x^2)$ and use the standard result $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+c$

$= \frac{1}{49}\left(\frac{1}{\left(\frac{2}{7}\right)}\arctan\left(\frac{x}{\left(\frac{2}{7}\right)}\right)\right)+c$ **M1**

$= \frac{1}{49}\left(\frac{7}{2}\arctan\left(\frac{7x}{2}\right)\right)+c = \frac{1}{14}\arctan\left(\frac{7x}{2}\right)+c$, i.e. $A=\frac{1}{14}$ and $B=\frac{7}{2}$ **A1** **[4 Marks]**

2. **By considering their behaviour at each limit, show that the following improper integrals diverge:**

a) $\int_4^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_4^t x^{-\frac{1}{2}}\,dx = \lim_{t\to\infty}\left[2x^{\frac{1}{2}}\right]_4^t$ **M1** ◄

Technique: In both part a) and part b) the integrand is undefined at one limit, so replace that limit with t and consider what happens as t tends to it

$= \lim_{t\to\infty}\left(2\sqrt{t}-4\right)$ **A1**

$\sqrt{t}\to\infty$ as $t\to\infty$, so $\int_4^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent **A1**

b) $\int_0^{\frac{\pi}{2}} \tan x\,dx = \lim_{t\to\frac{\pi}{2}}\int_0^t \tan x\,dx = \lim_{t\to\frac{\pi}{2}}\left[\ln|\sec x|\right]_0^t$ **M1**

$= \lim_{t\to\frac{\pi}{2}}\left(\ln|\sec t| - \ln|\sec 0|\right) = \lim_{t\to\frac{\pi}{2}}\left(\ln|\sec t|\right)$ **A1**

$\sec t\to\infty$ as $t\to\frac{\pi}{2}$, so $\ln|\sec t|\to\infty$ as $t\to\frac{\pi}{2}$, so $\int_0^{\frac{\pi}{2}} \tan x\,dx$ is divergent **A1** **[6 Marks]**

3. $x=\frac{1}{5}\sin\theta$, so $\frac{dx}{d\theta}=\frac{1}{5}\cos\theta \therefore dx = \frac{1}{5}\cos\theta\,d\theta$

$\therefore \int \frac{1}{\sqrt{1-25x^2}}\,dx = \int \frac{1}{\sqrt{1-25\left(\frac{1}{5}\sin\theta\right)^2}}\times\left(\frac{1}{5}\cos\theta\right)d\theta$ **M1** ◄

Technique: Substitute for x and use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to simplify

$= \int \frac{\frac{1}{5}\cos\theta}{\sqrt{1-25\times\frac{1}{25}\sin^2\theta}}\,d\theta = \int \frac{\frac{1}{5}\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$ **M1**

$= \int \frac{\frac{1}{5}\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \int \frac{\frac{1}{5}\cos\theta}{\cos\theta}\,d\theta = \int \frac{1}{5}\,d\theta = \frac{1}{5}\theta + c$ **M1**

Since $x=\frac{1}{5}\sin\theta$, $5x=\sin\theta$, so $\theta=\arcsin(5x)$ ◄

Hint: Remember to rewrite the answer in terms of x at the end

$\therefore \int \frac{1}{\sqrt{1-25x^2}}\,dx = \frac{1}{5}\theta + c = \frac{1}{5}\arcsin(5x)+c$, i.e. $A=\frac{1}{5}$ and $B=5$ **A1** **[4 Marks]**

4. **Use implicit differentiation to show that $\frac{d}{dx}\arccos\left(\frac{1}{x}\right)=\frac{1}{x\sqrt{x^2-1}}$**

Let $y=\arccos\left(\frac{1}{x}\right)$, so $\cos y=\frac{1}{x}$ **M1**

$\therefore -\sin y\frac{dy}{dx}=-\frac{1}{x^2}$ **M1**

So $\frac{dy}{dx}=\frac{1}{x^2\sin y}=\frac{1}{x^2\sqrt{1-\cos^2 y}}=\frac{1}{x^2\sqrt{1-\left(\frac{1}{x}\right)^2}}$ **M1**

Technique: Use the identity $\cos^2 y+\sin^2 y\equiv 1$ and the fact that sin y is positive in the range of arccos to rewrite the denominator in terms of x

$=\frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}=\frac{1}{x^2\times\frac{1}{x}\sqrt{x^2-1}}=\frac{1}{x\sqrt{x^2-1}}$ **A1** **[4 Marks]**

5. $\int_0^\infty 3x^2e^{-x^3}\,dx=\lim_{t\to\infty}\int_0^t 3x^2e^{-x^3}\,dx$ **M1**

Technique: The upper limit is infinity, so replace ∞ with t, complete the integration, and then take the limit as t tends to ∞

$=\lim_{t\to\infty}\left[-e^{-x^3}\right]_0^t=\lim_{t\to\infty}\left(-e^{-t^3}+1\right)$ **M1**

$-e^{-t^3}\to 0$ as $t\to\infty$, so $\int_0^\infty 3x^2e^{-x^3}\,dx$ converges to 1 **A1** **[3 Marks]**

6. a) $\int_0^{\frac{\pi}{2}}\frac{\cos x}{(\sin x+1)^2}\,dx=-\left[\frac{1}{\sin x+1}\right]_0^{\frac{\pi}{2}}$ **M1**

Technique: Notice that the integrand is of the form $kf'(x)(f(x))^n$

$=-\frac{1}{\sin\frac{\pi}{2}+1}+\frac{1}{\sin 0+1}$ **M1**

$=-\frac{1}{2}+1=\frac{1}{2}$ **A1**

Technique: The mean value of f(x) over the interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$

b) The mean value of $f(x)$ on $\left[0,\frac{\pi}{2}\right]$ is $\frac{1}{\pi/2-0}\int_0^{\frac{\pi}{2}}\frac{\cos x}{(\sin x+1)^2}\,dx$ **M1**

$=\frac{1}{\pi/2}\times\frac{1}{2}=\frac{2}{\pi}\times\frac{1}{2}=\frac{1}{\pi}$ **A1**

Alternatively: Recalculate the mean by integrating f(x) + 3 from 0 to π/2

c) Every value of $f(x)$ has 3 added to it, so the mean also increases by 3

So the mean of $f(x)+3$ on $\left[0,\frac{\pi}{2}\right]$ is $\frac{1}{\pi}+3\left(=\frac{3\pi+1}{\pi}\right)$ **A1** **[6 Marks]**

7. Let $\frac{20x-15}{(x^2+9)(x+6)} \equiv \frac{Ax+B}{x^2+9}+\frac{C}{x+6}$, so $(Ax+B)(x+6)+C(x^2+9) \equiv 20x-15$

$\therefore Ax^2+6Ax+Bx+6B+Cx^2+9C \equiv 20x-15$

Equate x^2 coefficients: $A + C = 0 \therefore A = -C$ **(1)**

Equate x coefficients: $6A + B = 20 \therefore B = 20 - 6A$ **(2)**

Equate constant terms: $6B + 9C = -15$ **(3) M1**

Substitute **(1)** into **(2)**:

$B = 20 - 6(-C) = 20 + 6C$ **(4)**

Substitute **(4)** into **(3)**: $6(20 + 6C) + 9C = -15$

$120 + 36C + 9C = -15$

$45C = -135$

$\therefore C = -3$

Using **(1)**, $A = -(-3) = 3$

Using **(2)**, $B = 20 - 6 \times 3 = 2$ **A1**

> **Alternatively:** To avoid getting three simultaneous equations, let $x = -6$ to begin with to get $C = -3$, and then either equate coefficients to get two simultaneous equations for A and B, or continue using substitution

> **Technique:** Integrate each fraction separately, and then use the laws of logarithms to combine the log terms

So $\int \frac{20x-15}{(x^2+9)(x+6)}\,dx = \int\left(\frac{3x+2}{x^2+9}-\frac{3}{x+6}\right)dx = \int \frac{3x}{x^2+9}\,dx + \int \frac{2}{x^2+9}\,dx - \int \frac{3}{x+6}\,dx$ **M1**

$= \frac{3}{2}\ln|x^2+9| + \frac{2}{3}\arctan\left(\frac{x}{3}\right) - 3\ln|x+6| + c$ **M1**

$= 3\left(\frac{1}{2}\ln|x^2+9| - \ln|x+6|\right) + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + c = 3\ln\left|\frac{\sqrt{x^2+9}}{x+6}\right| + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + c,$

i.e. $P = 3$ and $Q = \frac{2}{3}$ **A1** **[5 Marks]**

8. $2y^2 = x\cos\left(\frac{x}{2}\right) + x$

a) The x-coordinates of A and B are found by solving $0 = x\cos\left(\frac{x}{2}\right) + x$, i.e. $x\left(\cos\left(\frac{x}{2}\right)+1\right) = 0$

So either $x = 0$ **A1**

or $\cos\left(\frac{x}{2}\right) + 1 = 0 \therefore x = 2\pi$, i.e. x-coordinate of A is 0, and of B is 2π **A1**

> **Technique:** If the curve with equation $y = f(x)$ is rotated 2π radians about the x-axis between $x = a$ and $x = b$, then the volume of the solid generated is given by $V = \pi\int_a^b y^2\,dx = \pi\int_a^b (f(x))^2\,dx$

> **Technique:** Use integration by parts for the first integral

b) $y^2 = \frac{1}{2}x\cos\left(\frac{x}{2}\right) + \frac{1}{2}x$

$\therefore V = \pi\int_0^{2\pi}\left(\frac{1}{2}x\cos\left(\frac{x}{2}\right)+\frac{1}{2}x\right)dx = \frac{\pi}{2}\int_0^{2\pi} x\cos\left(\frac{x}{2}\right)\,dx + \frac{\pi}{2}\int_0^{2\pi} x\,dx$ **M1**

For the first integral, let $u = x$ and $\frac{dv}{dx} = \cos\left(\frac{x}{2}\right)$, so $\frac{du}{dx} = 1$ and $v = 2\sin\left(\frac{x}{2}\right)$

$\therefore \frac{\pi}{2}\int_0^{2\pi} x\cos\left(\frac{x}{2}\right)\,dx + \frac{\pi}{2}\int_0^{2\pi} x\,dx = \frac{\pi}{2}\left[2x\sin\left(\frac{x}{2}\right)\right]_0^{2\pi} - \frac{\pi}{2}\int_0^{2\pi} 2\sin\left(\frac{x}{2}\right)\,dx + \frac{\pi}{2}\left[\frac{1}{2}x^2\right]_0^{2\pi}$ **M1**

$= \frac{\pi}{2}(4\pi\sin\pi - 2\times0\times\sin 0) - \frac{\pi}{2}\left[-4\cos\left(\frac{x}{2}\right)\right]_0^{2\pi} + \frac{\pi}{2}\left(\frac{1}{2}(2\pi)^2 - \frac{1}{2}\times 0^2\right)$ **M1**

$= 0 - \frac{\pi}{2}(-4\cos\pi + 4\cos 0) + \pi^3$ **M1**

$= -2\pi - 2\pi + \pi^3 = \pi^3 - 4\pi$ **A1** **[7 Marks]**

9. $x = \frac{400}{11+t}$ **(1)**

$y = \frac{1}{2}t + 3 \therefore t = 2y - 6$ **(2)** **M1**

Substitute **(2)** into **(1)**:

$x = \frac{400}{11+2y-6} = \frac{400}{5+2y}$ **M1**

So $V = \pi \int_0^a \left(\frac{400}{5+2y}\right)^2 \, dy = \pi \int_0^a \frac{160\,000}{(5+2y)^2} \, dy$ **M1**

Try $\frac{1}{5+2y}$ so $\frac{d}{dy}\left(\frac{1}{5+2y}\right) = -\frac{2}{(5+2y)^2}$ **M1**

So $\pi \int_0^a \frac{160\,000}{(5+2y)^2} \, dy = \pi \left[-\frac{80\,000}{5+2y}\right]_0^a$ **M1**

$= \pi\left(-\frac{80\,000}{5+2a} + \frac{80\,000}{5+0}\right) = \pi\left(16\,000 - \frac{80\,000}{5+2a}\right)$ **M1**

We know that the volume is $15\,500\pi$, so $16\,000\pi - \frac{80\,000}{5+2a}\pi = 15\,500\pi$ **M1**

$16\,000 - 15\,500 = \frac{80\,000}{5+2a}$

$500(5+2a) = 80\,000$

$5 + 2a = 160$

$2a = 155 \therefore a = 77.5$ **A1** **[8 Marks]**

Technique: Convert the parametric equations to Cartesian form and use $V = \pi\int_a^b x^2 \, dy$ to find the volume of the solid generated when R is rotated 2π radians about the y-axis

Alternatively: Work with the equations in parametric form and remember to convert the limits to be in terms of t

TOTAL 47 MARKS

> **Tip:** To convert a polar equation into a Cartesian one, manipulate it until it is composed of terms involving r^2, $r\cos\theta$ and/or $r\sin\theta$. Then use the substitutions $r^2 = x^2 + y^2$, $r\cos\theta = x$ and $r\sin\theta = y$.

1. a) $r = 3$

$\therefore r^2 = 9$

$\therefore x^2 + y^2 = 9$ **M1A1**

b) $r\sin\theta = 1$

$\therefore y = 1$ **M1A1**

c) $r - 2\sec\theta = 0$

$\therefore r = 2\sec\theta$

$\therefore r\cos\theta = 2$ **M1**

$\therefore x = 2$ **A1**

d) $r\tan\theta - r = 0$

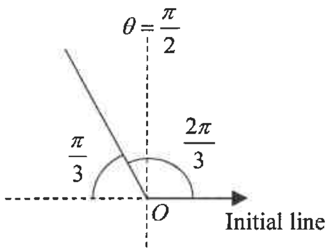
$\therefore r\dfrac{\sin\theta}{\cos\theta} = r$

$\therefore r\sin\theta = r\cos\theta$ **M1**

$\therefore y = x$ **A1**

[8 Marks]

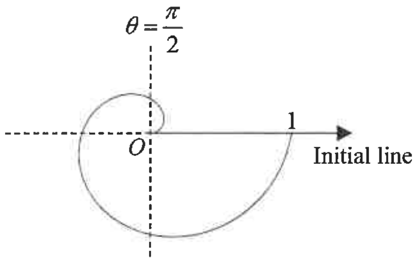
2. a) $\theta = \dfrac{2\pi}{3}$



For a half-line starting at the origin **A1**

For a line that makes an angle of $\dfrac{\pi}{3}$ with the negative x-axis **A1**

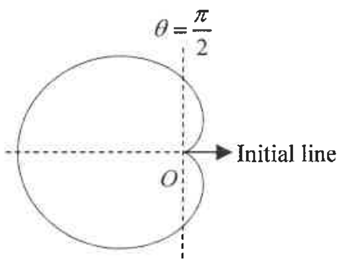
b) $r = \dfrac{\theta}{2\pi}$



For a spiral that increases in radius as θ increases **A1**

For a curve starting at the origin **A1**

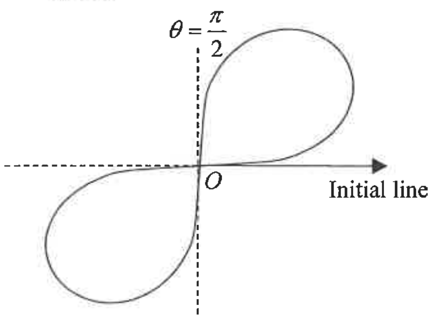
c) $r = 1 - \cos\theta$



For cardioid shape **A1**

For correct orientation **A1**

d) $r^2 = \sin 2\theta$



For figure-of-eight shape **A1**

For correct orientation **A1**

[8 Marks]

3. $r = 1 - \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

a) $y = r\sin\theta = (1 - \sin\theta)\sin\theta = \sin\theta - \sin^2\theta$

$\frac{dy}{d\theta} = \cos\theta - 2\sin\theta\cos\theta = \cos\theta(1 - 2\sin\theta)$ **M1**

So $\frac{dy}{d\theta} = 0$ when $\cos\theta = 0$ or when $1 - 2\sin\theta = 0$ **M1**

> **Tip:** The curve is parallel to the initial line when $\frac{dy}{d\theta} = 0$

$\cos\theta$ is never zero in the domain considered, so we are left with $1 - 2\sin\theta = 0$

$\therefore 2\sin\theta = 1$

$\therefore \sin\theta = \frac{1}{2}$

$\therefore \theta = \arcsin\left(\frac{1}{2}\right)$ **M1**

We want the solution in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$

and so $r = 1 - \sin\frac{\pi}{6} = 1 - \frac{1}{2} = \frac{1}{2}$

So the tangent to the curve is parallel to the initial line at the point with polar coordinates $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ **A1**

b) $x = r\cos\theta = (1 - \sin\theta)\cos\theta = \cos\theta - \cos\theta\sin\theta$

$\frac{dx}{d\theta} = -\sin\theta + \sin^2\theta - \cos^2\theta$ **M1**

$= -\sin\theta + \sin^2\theta - (1 - \sin^2\theta)$

$= 2\sin^2\theta - \sin\theta - 1$

So $\frac{dx}{d\theta} = 0$ when $2\sin^2\theta - \sin\theta - 1 = 0$ **M1**

> **Tip:** The curve is perpendicular to the initial line when $\frac{dx}{d\theta} = 0$

This is a quadratic equation in $\sin\theta$ that factorises as:

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$\therefore \sin\theta = -\frac{1}{2}$ or 1 **M1**

We want the solution in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\sin\theta \neq 1$ in this interval, so:

$\theta = \arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

and so $r = 1 - \sin\left(-\frac{\pi}{6}\right) = 1 + \frac{1}{2} = \frac{3}{2}$

So the tangent to the curve is perpendicular to the initial line at the point with polar coordinates $\left(\frac{3}{2}, -\frac{\pi}{6}\right)$ **A1**

[8 Marks]

4. $r = \sqrt{\theta}$

The region R is enclosed by the curve between $\theta = 0$ and $\theta = \frac{3\pi}{2}$, so:

$\text{Area} = \frac{1}{2}\int_0^{\frac{3\pi}{2}} \left(\sqrt{\theta}\right)^2 d\theta$ **M1M1**

> **Tip:** The area enclosed by the curve $r = f(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is given by the formula $\frac{1}{2}\int_\alpha^\beta r^2\, d\theta$, or $\frac{1}{2}\int_\alpha^\beta (f(\theta))^2\, d\theta$

$= \frac{1}{2}\int_0^{\frac{3\pi}{2}} \theta\, d\theta$

$= \frac{1}{2}\left[\frac{1}{2}\theta^2\right]_0^{\frac{3\pi}{2}}$ **M1**

$= \frac{1}{2}\left(\frac{1}{2} \times \left(\frac{3\pi}{2}\right)^2 - \frac{1}{2} \times 0\right)$

$= \frac{9}{16}\pi^2$ **A1**

[4 Marks]

5. a) $x^2 + 2y^2 = 3$

$\therefore x^2 + y^2 + y^2 = 3$

$\therefore r^2 + (r\sin\theta)^2 = 3$ **M1**

$\therefore r^2(1+\sin^2\theta) = 3$

$\therefore r^2 = \dfrac{3}{1+\sin^2\theta}$ **A1**

b) $y = 2x + 1$

$\therefore r\sin\theta = 2r\cos\theta + 1$ **M1**

$\therefore r\sin\theta - 2r\cos\theta = 1$

$\therefore r(\sin\theta - 2\cos\theta) = 1$ **M1**

$\therefore r = \dfrac{1}{\sin\theta - 2\cos\theta}$ **A1**

c) $x^2 - y^2 = 1$

$\therefore r^2\cos^2\theta - r^2\sin^2\theta = 1$ **M1**

$\therefore r^2(\cos^2\theta - \sin^2\theta) = 1$ ◄

Tip: The question asks for a single trigonometric function, so you need to turn $\cos^2\theta - \sin^2\theta$ into one function. In this case the double angle formula for cosine helps.

$\therefore r^2\cos 2\theta = 1$ **M1**

$\therefore r^2 = \sec 2\theta$ **A1**

[8 Marks]

6. $r = a - \sin\theta$

Area $= \dfrac{1}{2}\int_0^{2\pi} (a-\sin\theta)^2 \, d\theta$ **M1**

$= \dfrac{1}{2}\int_0^{2\pi} (a^2 - 2a\sin\theta + \sin^2\theta)\, d\theta$ ◄

Tip: Rearrange the double angle formula for cosine to write $\sin^2\theta = \dfrac{1}{2} - \dfrac{1}{2}\cos 2\theta$

$= \dfrac{1}{2}\int_0^{2\pi} \left(a^2 - 2a\sin\theta + \dfrac{1}{2} - \dfrac{1}{2}\cos 2\theta\right) d\theta$

$= \dfrac{1}{2}\left[a^2\theta + 2a\cos\theta + \dfrac{1}{2}\theta - \dfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$ **M1**

$= \dfrac{1}{2}\left(a^2 \times 2\pi + 2a\cos 2\pi + \dfrac{1}{2}\times 2\pi - \dfrac{1}{4}\sin 4\pi - a^2 \times 0 - 2a\cos 0 - \dfrac{1}{2}\times 0 + \dfrac{1}{4}\sin 0\right)$

$= \dfrac{1}{2}(2\pi a^2 + 2a + \pi - 2a)$

$= \left(a^2 + \dfrac{1}{2}\right)\pi$ **A1**

We are told in the question that this area is $\dfrac{33}{2}\pi$, so $a^2 + \dfrac{1}{2} = \dfrac{33}{2}$ **M1**

$\therefore a^2 = \dfrac{33}{2} - \dfrac{1}{2} = 16$

So $a = \pm\sqrt{16} = \pm 4$, but we are told in the question that $a > 1$, so $a = 4$ **A1** **[5 Marks]**

7. $r = 1 + \cos\theta$

$y = r\sin\theta = (1 + \cos\theta)\sin\theta$

$\frac{dy}{d\theta} = -\sin^2\theta + (1+\cos\theta)\cos\theta$ **M1**

$= \cos\theta + \cos^2\theta - \sin^2\theta$

$= \cos\theta + \cos^2\theta - (1-\cos^2\theta)$

$= 2\cos^2\theta + \cos\theta - 1$

So $\frac{dy}{d\theta} = 0$ when $2\cos^2\theta + \cos\theta - 1 = 0$ **M1**

$\therefore (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\therefore 2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\therefore \theta = \arccos\left(\frac{1}{2}\right)$ or $\theta = \arccos(-1)$ **M1**

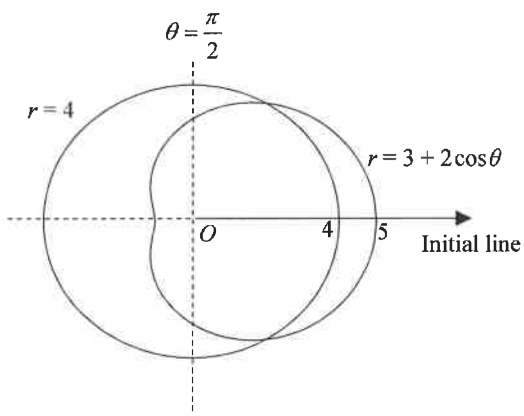
We are only considering $0 \le \theta \le \frac{\pi}{2}$, so the only solution we have is $\theta = \frac{\pi}{3}$

and so $r = 1 + \cos\frac{\pi}{3} = \frac{3}{2}$ **A1**

So the tangent passes through the point $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and is parallel to the initial line, hence has Cartesian equation

$y = r\sin\theta = \frac{3}{2} \times \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}$ **A1** **[5 Marks]**

8. a)



For curve with 'dimple' **A1**
For correct orientation of dimpled curve **A1**
For circle centred at the origin **A1**
For crossing points in the correct quadrants **A1**

b) The curves cross where $3 + 2\cos\theta = 4$ **M1**

$\therefore 2\cos\theta = 1$

$\therefore \theta = \arccos\left(\frac{1}{2}\right)$ **M1**

So for $0 \le \theta \le 2\pi$, $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$ **A1**

Since both points lie on the curve $r = 4$, the coordinates of the crossing points are $\left(4, \frac{\pi}{3}\right)$ and $\left(4, \frac{5\pi}{3}\right)$ **A1**

[8 Marks]

TOTAL 54 MARKS

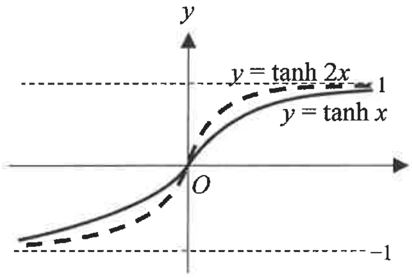
1. $\cosh(\ln 2) = \dfrac{e^{\ln 2} + e^{-\ln 2}}{2}$ **M1**

Hint: Use the power law of logarithms, $a\log b = \log b^a$, to convert $-\ln 2$ to $\ln\dfrac{1}{2}$

$= \dfrac{2 + \frac{1}{2}}{2}$ **M1**

$= \dfrac{\frac{5}{2}}{2} = \dfrac{5}{4}$ **A1** **[3 Marks]**

2. a)



A1A1

Hint: $y = \tanh 2x$ is a stretch of $y = \tanh x$ by scale factor 1/2 parallel to the x-direction

Asymptotes at $y = -1$ and $y = 1$ **A1**

b) $-1 < f(x) < 1$ **A1** **[4 Marks]**

3. a) $\displaystyle\int \frac{2}{\sqrt{x^2-1}}\,dx = 2\int \frac{1}{\sqrt{x^2-1}}\,dx = 2\operatorname{arcosh} x + c$ **A1**

Technique: Integrals of this form can be found in your formula book

b) $\displaystyle\int \frac{5}{\sqrt{4+x^2}}\,dx = 5\int \frac{1}{\sqrt{4+x^2}}\,dx = 5\operatorname{arsinh}\left(\frac{x}{2}\right) + c$ **A1** **[2 Marks]**

4. a) $\dfrac{d}{dx}\sinh 2x = 2\cosh 2x$ **A1**

Technique: Use the chain rule

b) $\dfrac{d}{dx}\sinh 2x\cosh x = 2\cosh 2x\cosh x + \sinh 2x\sinh x$ **M1**

Technique: Use the product rule with the chain rule

$\therefore \dfrac{d^2}{dx^2}\sinh 2x\cosh x = (2\times 2\sinh 2x\times\cosh x + 2\cosh 2x\times\sinh x) + (2\cosh 2x\times\sinh x + \sinh 2x\times\cosh x)$ **M1M1**

$= (4\sinh 2x\cosh x + \sinh 2x\cosh x) + (2\cosh 2x\sinh x + 2\cosh 2x\sinh x)$ **M1**

$= 5\sinh 2x\cosh x + 4\cosh 2x\sinh x$ **A1** **[6 Marks]**

5. **Show that $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$ for $x \neq 0$**

$LHS = \coth^2 x - 1 \equiv \dfrac{\cosh^2 x}{\sinh^2 x} - 1$

Alternatively: Start with $\cosh^2 x - \sinh^2 x \equiv 1$ and divide by $\sinh^2 x$

$\equiv \dfrac{\cosh^2 x - \sinh^2 x}{\sinh^2 x}$ **M1**

$\equiv \dfrac{1}{\sinh^2 x}$ **M1**

$\equiv \operatorname{cosech}^2 x = RHS$ **A1** **[3 Marks]**

6. a) $\displaystyle\int \sinh 4x\,dx = \int\left(\frac{1}{4}\times 4\sinh 4x\right)dx = \frac{1}{4}\cosh 4x + c \quad \left(\text{or } \frac{1}{8}\left(e^{4x}+e^{-4x}\right)+c\right)$ **M1A1**

Alternatively: Write $\sinh 4x$ in exponential form first before integrating

b) $\displaystyle\int 6e^{-x}\sinh 4x\,dx = 6\int e^{-x}\left(\frac{e^{4x}-e^{-4x}}{2}\right)dx$ **M1**

$= 3\displaystyle\int\left(e^{3x}-e^{-5x}\right)dx$ **M1**

$= 3\left(\dfrac{1}{3}e^{3x} + \dfrac{1}{5}e^{-5x}\right) + c = e^{3x} + \dfrac{3}{5}e^{-5x} + c$ **A1** **[5 Marks]**

7. $6\cosh x - 2\sinh x = 9$

$$6\left(\frac{e^x + e^{-x}}{2}\right) - 2\left(\frac{e^x - e^{-x}}{2}\right) = 9$$

$3e^x + 3e^{-x} - e^x + e^{-x} - 9 = 0$ **M1**

$2e^x - 9 + 4e^{-x} = 0$ **M1**

$2e^{2x} - 9e^x + 4 = 0$ ←

$(2e^x - 1)(e^x - 4) = 0$ **M1**

So $e^x = \frac{1}{2}$ or $e^x = 4$ **M1**

So $x = \ln\frac{1}{2}$ or $x = \ln 4$ **A1A1** **[6 Marks]**

Technique: Rewrite the equation in exponential form then multiply by e^x to get a quadratic equation in e^x, and factorise to solve

8. a) **Show that $y = \ln\left(x + \sqrt{x^2 + 1}\right)$**

$\text{arsinh}\, x = y$, so $x = \sinh y = \frac{e^y - e^{-y}}{2}$ **M1** ←

$2x = e^y - e^{-y}$

$e^{2y} - 2xe^y - 1 = 0$ **M1**

$(e^y - x)^2 - x^2 - 1 = 0$ **M1**

$e^y - x = \pm\sqrt{x^2 + 1}$

$e^y = x \pm \sqrt{x^2 + 1} = x + \sqrt{x^2 + 1}$ **M1** ←

$\therefore y = \ln\left(x + \sqrt{x^2 + 1}\right)$ **A1**

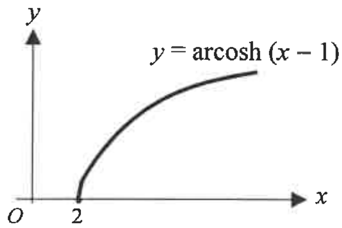
Technique: Rearrange and rewrite the equation in exponential form, and then multiply by e^y to get a quadratic equation in e^y. Then complete the square to solve.

Hint: You can ignore $x - \sqrt{x^2 + 1}$ because $\sqrt{x^2 + 1} > x$, so e^y would be negative for this solution

b) $\text{arsinh}\, 2 = \ln\left(2 + \sqrt{2^2 + 1}\right)$ **M1**

$= \ln\left(2 + \sqrt{4 + 1}\right) = \ln\left(2 + \sqrt{5}\right)$ **A1** **[7 Marks]**

9.



Hint: $y = \text{arcosh}\,(x - 1)$ is a translation of $y = \text{arcosh}\, x$ 1 unit to the right

A1 correct shape (non-negative)

A1 x-intercept at $x = 2$ **[2 Marks]**

10. **Prove that $\cosh^2 x + \sinh^2 x \equiv \cosh 2x$**

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \quad \textbf{M1}$$

$$= \left(\frac{1}{2}\right)^2 \times \left(e^{2x} + e^{-2x} + 2\right) + \left(\frac{1}{2}\right)^2 \times \left(e^{2x} + e^{-2x} - 2\right) \quad \textbf{M1}$$ ←

$$= \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} + \frac{1}{2} + \frac{1}{4}e^{2x} + \frac{1}{4}e^{-2x} - \frac{1}{2} = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$$

$$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x, \text{ as required} \quad \textbf{A1}$$ **[3 Marks]**

Technique: Expand the left-hand side in exponential form and show that it is equivalent to the exponential form of the right-hand side

11. a) $\cosh^2 x - \sinh^2 x \equiv 1 \therefore \sinh x \equiv \pm\sqrt{\cosh^2 x - 1}$

Since $\cosh x = \frac{3}{2}$, $\sinh x = \pm\sqrt{\left(\frac{3}{2}\right)^2 - 1}$ **M1**

$\therefore \sinh x = \pm\sqrt{\frac{9}{4} - 1} = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$ **M1**

So $\tanh x = \frac{\sinh x}{\cosh x} = \pm\frac{\sqrt{5}/2}{3/2} = \pm\frac{\sqrt{5}}{3}$ **A1**

Technique: To find tanh x, first find sinh x using the identity $\cosh^2 x - \sinh^2 x \equiv 1$

Alternatively: Use the identity $1 - \tanh^2 x \equiv \text{sech}^2 x$ with sech $x = 2/3$

b) $\cosh x = \frac{3}{2}$, so $\frac{e^x + e^{-x}}{2} = \frac{3}{2} \therefore e^x + e^{-x} = 3$

$e^{2x} - 3e^x + 1 = 0$ **M1**

Technique: Multiply by e^x and solve using the quadratic formula

Using the quadratic formula with $a = 1$, $b = -3$ and $c = 1$:

$e^x = \frac{3 \pm \sqrt{(-3)^2 - 4\times 1\times 1}}{2\times 1}$ **M1**

$= \frac{3 \pm \sqrt{5}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$ **M1**

So $x = \ln\left(\frac{3}{2} \pm \frac{\sqrt{5}}{2}\right) = 0.962423\ldots$ or $-0.962423\ldots = \pm 0.96$ (2 d.p.) **A1A1** **[8 Marks]**

12. $y = 12\cosh x - 6\sinh x$

$\frac{dy}{dx} = 12\sinh x - 6\cosh x$ **M1**

The stationary point is where $\frac{dy}{dx} = 0$

$\therefore 12\sinh x - 6\cosh x = 0$

Alternatively: Rearrange this to tanh x = ½ and use the logarithmic definition of artanh to find x

So $12\left(\frac{e^x - e^{-x}}{2}\right) - 6\left(\frac{e^x + e^{-x}}{2}\right) = 0$ **M1**

$6e^x - 6e^{-x} = 3e^x + 3e^{-x}$

$3e^x = 9e^{-x}$

$e^{2x} = 3 \therefore e^x = \sqrt{3}$ **M1**

Technique: Ignore the alternative solution $e^x = -\sqrt{3}$ since e^x cannot be negative

So $x = \ln\left(\sqrt{3}\right)$ **A1**

$y = 12\left(\frac{e^x + e^{-x}}{2}\right) - 6\left(\frac{e^x - e^{-x}}{2}\right)$

$= 6e^x + 6e^{-x} - 3e^x + 3e^{-x} = 3e^x + 9e^{-x}$ **M1**

Using $x = \ln\left(\sqrt{3}\right)$:

$y = 3e^{\ln\sqrt{3}} + 9e^{-\ln\sqrt{3}} = 3\sqrt{3} + \frac{9}{\sqrt{3}} = 6\sqrt{3}$ **A1**

So the coordinates of the stationary point are $\left(\ln\left(\sqrt{3}\right), 6\sqrt{3}\right)$ (also accept $\frac{1}{2}\ln 3$ for $\ln\left(\sqrt{3}\right)$) **[6 Marks]**

13. $\int \frac{1}{\sqrt{4x^2-16x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{4\left(x^2-4x\right)}}\,\mathrm{d}x$ **M1**

Technique: Take out a factor of 4 in the denominator, and then complete the square

$$= \int \frac{1}{\sqrt{4\left(\left(x-2\right)^2-4\right)}}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{\left(x-2\right)^2-4}}\,\mathrm{d}x \quad \textbf{M1}$$

Let $u = x-2$ so $\mathrm{d}x = \mathrm{d}u$

$$\therefore \frac{1}{2}\int \frac{1}{\sqrt{\left(x-2\right)^2-4}}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{\sqrt{u^2-4}}\,\mathrm{d}u \quad \textbf{M1}$$

$$= \frac{1}{2}\operatorname{arcosh}\left(\frac{u}{2}\right)+c \quad \textbf{M1}$$

Technique: Use the standard result $\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{x}{a}\right)+c$

$$= \frac{1}{2}\operatorname{arcosh}\left(\frac{x-2}{2}\right)+c \quad \textbf{A1}$$

[5 Marks]

14. Using the substitution $x = \frac{1}{2}\left(1+5\cosh u\right)$, $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{5}{2}\sinh u$ so $\mathrm{d}x = \frac{5}{2}\sinh u\,\mathrm{d}u$ **M1**

$$\therefore x^2-x-6 = \left(\frac{1}{2}\left(1+5\cosh u\right)\right)^2 - \left(\frac{1}{2}\left(1+5\cosh u\right)\right)-6$$

Alternatively: Complete the square of the expression under the square root. Then rearrange the substitution to get $x - ½ = \frac{5}{2}\cosh u$, and substitute for $x - ½$.

$$= \frac{1}{4}\left(1+10\cosh u+25\cosh^2 u\right)-\frac{1}{2}\left(1+5\cosh u\right)-6$$

$$= \frac{1}{4}+\frac{5}{2}\cosh u+\frac{25}{4}\cosh^2 u-\frac{1}{2}-\frac{5}{2}\cosh u-6$$

$$= \frac{25}{4}\cosh^2 u-\frac{25}{4} \quad \textbf{M1}$$

$$= \frac{25}{4}\sinh^2 u \quad \textbf{M1}$$

Technique: Use the identity $\cosh^2 u - \sinh^2 u \equiv 1$ to rewrite $x^2 - x - 6$ in terms of $\sinh u$

$$\text{So } \int \frac{1}{\sqrt{x^2-x-6}}\,\mathrm{d}x = \int \frac{1}{\sqrt{25/4\,\sinh^2 u}} \times \frac{5}{2}\sinh u\,\mathrm{d}u = \int \frac{\frac{5}{2}\sinh u}{\frac{5}{2}\sinh u}\,\mathrm{d}u \quad \textbf{M1}$$

$$= \int 1\,\mathrm{d}u = u+c \quad \textbf{M1}$$

$x = \frac{1}{2}\left(1+5\cosh u\right) \therefore \cosh u = \frac{2x-1}{5}$ and so $u = \operatorname{arcosh}\left(\frac{2x-1}{5}\right)$

$$\text{So } \int \frac{1}{\sqrt{x^2-x-6}}\,\mathrm{d}x = \operatorname{arcosh}\left(\frac{2x-1}{5}\right)+c \quad \textbf{A1}$$

[6 Marks]

TOTAL 66 MARKS

1. $\frac{dy}{dx} + \frac{3y}{x} = 4 - \frac{6}{x}$

We are told in the question that the integrating factor is x^3. So, multiplying through by this:

$x^3 \frac{dy}{dx} + 3x^2 y = 4x^3 - 6x^2$ **M1**

> **Technique:** Multiply through by the integrating factor x^3. The left-hand side is then the derivative of $x^3 y$.

$\therefore \frac{d}{dx}(x^3 y) = 4x^3 - 6x^2$

$\therefore x^3 y = \int (4x^3 - 6x^2)\,dx$ **M1**

$= x^4 - 2x^3 + c$ **A1**

So the general solution is $y = \frac{x^4 - 2x^3 + c}{x^3}$ **A1** **[4 Marks]**

2. a) i) $y'' + 9y' + 14y = 0$

> **Tip:** The auxiliary equation for $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$

The auxiliary equation is $m^2 + 9m + 14 = 0$ **M1**

$\therefore (m+2)(m+7) = 0$

So $m = -2$ or $m = -7$ **A1**

So the general solution is $y = Ae^{-2x} + Be^{-7x}$ **A1**

> **Tip:** If the auxiliary equation has distinct real roots α and β, then the general solution to the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$

ii) $y'' - 6y' + 9y = 0$

The auxiliary equation is $m^2 - 6m + 9 = 0$ **M1**

$\therefore (m-3)^2 = 0$

So $m = 3$ is a repeated root **A1**

So the general solution is $y = (A + Bx)e^{3x}$ **A1**

> **Tip:** If the auxiliary equation has a repeated real root α, then the general solution to the differential equation is $y = (A + Bx)e^{\alpha x}$

b) $y'' + 4y = 0$

The auxiliary equation is $m^2 + 4 = 0$ **M1**

So $m = \pm 2i$ **A1**

So the general solution is $y = A\cos 2x + B\sin 2x$ **A1**

> **Tip:** If the auxiliary equation has complex conjugate roots $p \pm qi$, then the general solution to the differential equation is $y = e^{px}(A\cos qx + B\sin qx)$

We are told that $y = 5$ when $x = 0$, so:

$5 = A\cos 0 + B\sin 0 = A$, so $A = 5$

We are also told that $y = -6$ when $x = \frac{\pi}{4}$, so, using $A = 5$:

$-6 = 5\cos\frac{\pi}{2} + B\sin\frac{\pi}{2} = B$, so $B = -6$ **M1**

So the particular solution is $y = 5\cos 2x - 6\sin 2x$ **A1** **[11 Marks]**

3. $\ddot{x} = -16x$

a) The particle is moving with simple harmonic motion **B1**

b) **Show that the general solution to the differential equation is $x = A\cos 4t + B\sin 4t$**

The auxiliary equation is $m^2 + 16 = 0$ **M1**

So $m = \pm 4i$ **A1**

So the general solution is $x = A\cos 4t + B\sin 4t$ **A1**

c) We are told that $x = -5$ when $t = 0$, so:

$-5 = A\cos 0 + B\sin 0 = A$ so $A = -5$ **M1**

We are also told that $\dot{x} = 12$ when $t = 0$

$\dot{x} = -4A\sin 4t + 4B\cos 4t$ **A1**

So:

$12 = -4A\sin 0 + 4B\cos 0 = 4B$, so $B = 3$

So the particular solution is $x = 3\sin 4t - 5\cos 4t$ **A1** **[7 Marks]**

4. $y'' - 4y' + 3y = 6x + 1$

a) The complementary function satisfies $y'' - 4y' + 3y = 0$

The auxiliary equation is $m^2 - 4m + 3 = 0$ **M1**

$\therefore (m-1)(m-3) = 0$

So $m = 1$ or $m = 3$ **A1**

So the complementary function is $y = Ae^x + Be^{3x}$ **A1**

> **Tip:** The complementary function for a non-homogeneous second-order differential equation is a solution to the homogeneous equation formed by setting the right-hand side equal to zero

b) The right-hand side is a linear function, so try the particular integral $y = \lambda + \mu x$, where $\lambda, \mu \in \mathbb{R}$

Then $y' = \mu$ and $y'' = 0$. So:

$y'' - 4y' + 3y = 0 - 4\mu + 3(\lambda + \mu x)$ **M1**

$= 3\mu x + 3\lambda - 4\mu$

So $3\mu x + 3\lambda - 4\mu = 6x + 1$ **M1**

Comparing coefficients of x gives $3\mu = 6$, so $\mu = 2$

Comparing constant coefficients and using $\mu = 2$ gives $3\lambda - 8 = 1$, so $\lambda = 3$ **M1**

and so the particular integral is $2x + 3$ **A1**

> **Tip:** The particular integral for a non-homogeneous second-order differential equation should be a function with the same form as the right-hand side of the equation, but with unknown coefficients

c) The general solution is the complementary function plus the particular integral, so:

$y = Ae^x + Be^{3x} + 2x + 3$ **A1** **[8 Marks]**

5. $\frac{dx}{dt} = x - \frac{5}{3}y$

$\frac{dy}{dt} = \frac{2}{3}x - y$

a) **Show that $\frac{d^2x}{dt^2} + \frac{x}{9} = 0$**

From the first differential equation, $y = \frac{3}{5}x - \frac{3}{5}\frac{dx}{dt}$ **A1**

So $\frac{dy}{dt} = \frac{3}{5}\frac{dx}{dt} - \frac{3}{5}\frac{d^2x}{dt^2}$ **A1**

Substituting these into the second differential equation gives:

$\frac{3}{5}\frac{dx}{dt} - \frac{3}{5}\frac{d^2x}{dt^2} = \frac{2}{3}x - \frac{3}{5}x + \frac{3}{5}\frac{dx}{dt}$ **M1**

$\therefore \frac{3}{5}\frac{d^2x}{dt^2} + \frac{1}{15}x = 0$

So $\frac{d^2x}{dt^2} + \frac{1}{9}x = 0$ **A1**

> **Technique:** Use the first equation to write y in terms of x and its derivative, and then differentiate this with respect to t to find $\frac{dy}{dt}$ in terms of x and its derivatives. You can then substitute these expressions into the second equation.

b) We need to solve $\frac{d^2x}{dt^2} + \frac{1}{9}x = 0$ to find x in terms of t

The auxiliary equation of this differential equation is $m^2 + \frac{1}{9} = 0$ **M1**

So $m = \pm\frac{1}{3}\mathrm{i}$ **A1**

So the general solution is $x = A\cos\frac{t}{3} + B\sin\frac{t}{3}$ **A1** **[7 Marks]**

6. a) $\frac{dy}{dx} - \frac{y}{x} = x$

The integrating factor is $e^{\int -\frac{1}{x}dx} = e^{-\ln x}$ **M1A1**

$= e^{\ln\frac{1}{x}}$

$= \frac{1}{x}$ **A1**

> **Technique:** For a differential equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$, start by multiplying through by the integrating factor $\exp\left(\int P(x)\,dx\right)$. The left-hand side is then the derivative of $y \times \exp\left(\int P(x)\,dx\right)$.

Multiplying through by this gives:

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$$

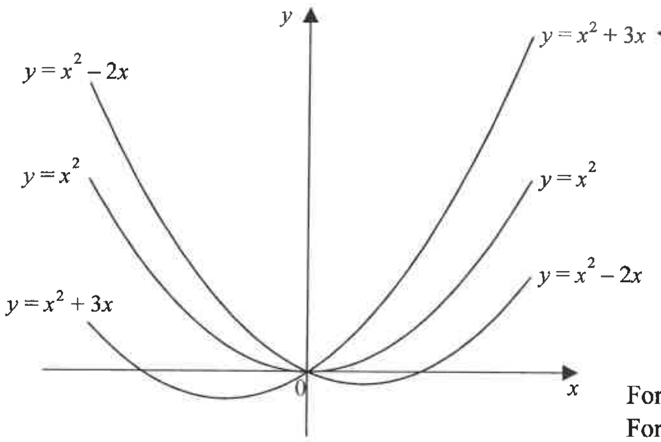
$$\therefore \frac{d}{dx}\left(\frac{y}{x}\right) = 1$$

$$\therefore \frac{y}{x} = \int 1 dx \quad \textbf{M1}$$

$$= x + c \quad \textbf{A1}$$

So the general solution is $y = x^2 + cx$ **A1**

b) $x^2 + cx = x(x + c)$, so the particular solutions are parabolas with roots at $x = 0$ and $x = -c$, e.g.:



Tip: Pick three small integer values for c and sketch the corresponding graphs of $y = x(x + c)$

For three correctly oriented parabolas **M1**
For curves that pass through the origin **M1** **[8 Marks]**

7. $\ddot{x} + 6\dot{x} + 9x = 0$

a) The auxiliary equation is $m^2 + 6m + 9 = 0$ **M1**

$\therefore (m+3)^2 = 0$

So the repeated root is $m = -3$ **A1**

So the general solution is $x = (A + Bt)e^{-3t}$ **A1**

We are told that $x = 15$ when $t = 0$, so:

$15 = (A + B \times 0)e^0 = A$ so $A = 15$

We are also told that $\dot{x} = 5$ when $t = 0$. Using $A = 15$,

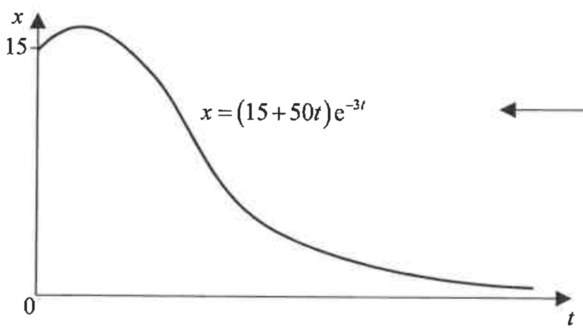
$\dot{x} = -3(15 + Bt)e^{-3t} + Be^{-3t}$ **A1**

So $5 = -3(15 + B \times 0)e^0 + Be^0 = -45 + B$, and so $B = 50$ **M1**

So the particular solution is $x = (15 + 50t)e^{-3t}$ **A1**

Technique: You are given a boundary condition for $\dot{x}$. Differentiate your general solution with the value of A to get an expression for $\dot{x}$ that involves B, and then use the boundary condition to find B.

b)



Tip: Find the value of x at some small values of t using your calculator to help determine the shape of the graph. Note that the gradient starts off positive since that is one of the initial conditions. Then find the value of x at large values of t to check for asymptotic behaviour.

For an initially increasing curve that is then decreasing **M1**
For an asymptote at $x = 0$ **M1**

c) Any suitable criticism **B1**

e.g.:

- Bethany is likely to oscillate around some point rather than asymptotically approach it **[9 Marks]**

TOTAL 54 MARKS

