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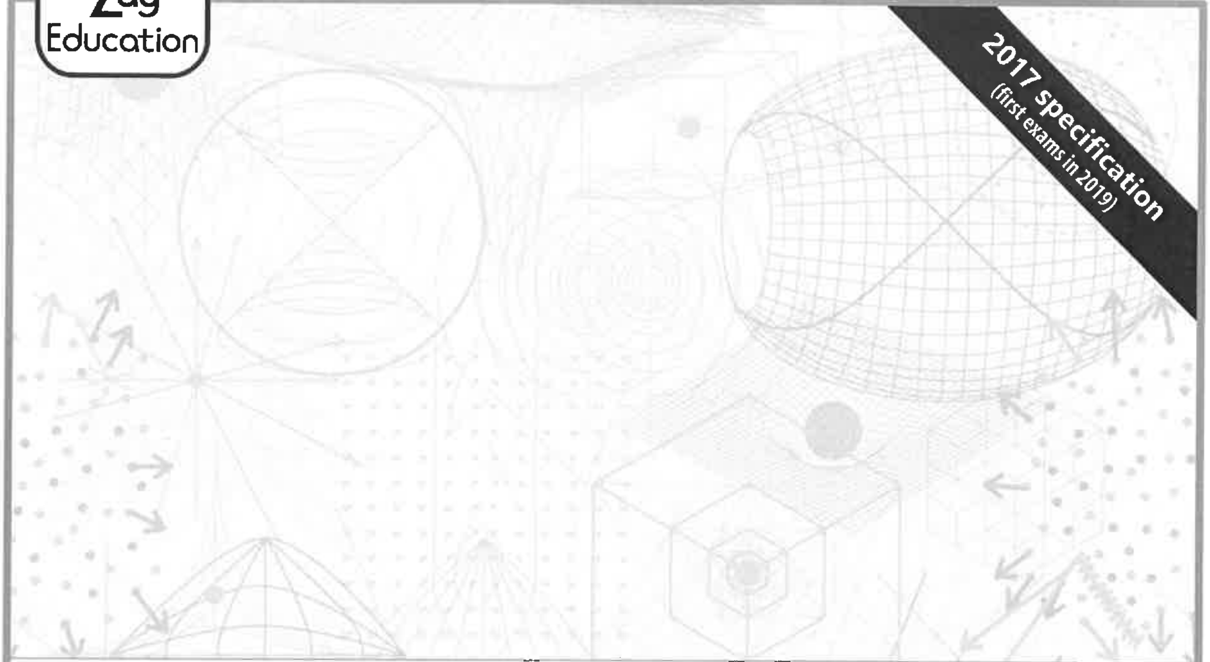

Topic Tests:
Challenge Tests – Set A

A Level Edexcel Further Mathematics: Core Pure Mathematics: Part 1#

#Every topic of AS (8FM0) Core Pure Mathematics

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POD 9426

Contents

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This resource is cross-referenced to the following textbook: the Pearson Education textbook *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* by Greg Attwood, Jack Barraclough, Ian Bettison, Lee Cope, Charles Garnet Cox, Daniel Goldberg, Alistair Macpherson, Bronwen Moran, Su Nicholson, Laurence Pateman, Joe Petran, Keith Pledger, Harry Smith, Geoff Staley and Dave Wilkins (ISBN 978-1292183336). ZigZag Education is not affiliated with Pearson Education in any way nor is this publication authorised by, associated with, sponsored by or endorsed by Pearson Education unless explicitly stated on the front cover of this publication.

Teacher's Introduction

Content

This pack contains 6 challenge level topic tests for A Level Edexcel Further Mathematics: Core Pure Mathematics.

Each test comes with fully worked solutions, containing tips, hints and technique boxes to help students on a particular question. Answers should be given to three significant figures unless specified in the question.

These topic tests have been **fully cross-referenced** to the Pearson textbook, *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* (ISBN 978-1292183336), for your convenience (see reference sheet on page 2). Each test has been designed to reflect the specification fully.

About the challenge tests

These **challenge** tests have been designed to **stretch and challenge** your students. 50% of the marks come from questions similar in style to our fundamentals tests. These questions isolate and test the core skills in each topic. The other 50% of the marks come from questions of increased difficulty that progress and start to combine the concepts in the topic.

Timings

The recommended times for students to complete each test are given at the top of individual tests.

Calculator use

The effective use of a calculator is one of the objectives of the new specification and is encouraged for all the enclosed tests.

Also available from ZigZag Education

The perfect starting point for students of all abilities are our **fundamentals** tests. These isolate and test the core skills in each topic so that your students can show what they can do. They get a confidence boost and you can see at a glance where each student's weaknesses lie.

To prepare students for the exam itself, our **expert** tests contain 25% repeated marks from the fundamentals and challenge tests, and 75% exam-style material with compound/multistep questions.

For each collection of Set A tests we also offer a corresponding collection of Set B duplicated tests with the same styles of questions but different numbers. This allows for a variety of **flexible** uses including:

- **Test → Homework**: Students use test B as a homework to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Homework → Test**: Students revise as homework using test A before doing test B in class under test conditions.
- **Test → Classwork**: Students work through test B with teacher input to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Classwork → Test:** Students work through test A with teacher input, before checking their learning by completing test B under test conditions.

For total flexibility, the Set A and Set B tests of all three levels can be run on a rolling basis, using the fundamentals tests as starters, with a time interval between them, leaving one expert level test to use at the end of the course for topic revision.

Cross-referencing Grid

Topic	Edexcel spec. points	Subtopics	Chapter Reference - Edexcel Pearson textbook [ISBN: 9781292183336]
Complex Numbers	2.1–2.7	Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram	1, 2
Algebra and Functions	4.1–4.3	Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots	3, 4
Volumes of Revolution	5.1	Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes	5
Matrices	3.1–3.8	Matrices, matrix multiplication, determinants, inverting 2×2 and 3×3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation	6, 7
Proof by Induction	1.1	Proof by mathematical induction, proving divisibility results, proving statements involving matrices	8
Further Vectors	6.1–6.5	Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars	9

Complex Numbers – Test A (40 mins)

Fundamentals ✓**Challenge** Expert

Subtopics: Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram

1. Given that $z_1 = -2+5\text{i}$ and $z_2 = 3-4\text{i}$, write each of the following in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:
   a) $z_2 - z_1$ b) $2z_1 + 4z_2$ c) $z_1 z_2$ d) z_2^* **[6]**

2. Express each of the following in the form $r(\cos\theta + \text{i}\sin\theta)$, where θ is in radians and $-\pi < \theta \le \pi$.
   Where appropriate, give θ either in terms of π or to **3 significant figures**, and r in **simplified surd form**.
   a) $5+12\text{i}$ b) $3\sqrt{3}-9\text{i}$ **[6]**

3. On separate Argand diagrams, **sketch the locus** of z and find its **Cartesian equation** for each of the following:
   a) $|z-3|=3$ b) $|z-2+\text{i}|=3$ **[6]**

4. a) Write each of the following in the form $a + b\text{i}$, where a and b are rational numbers in their lowest terms:
   i) $\dfrac{3+2\text{i}}{5-\text{i}}$ ii) $\dfrac{6+\text{i}}{2+\text{i}}$ iii) $\dfrac{2-\text{i}}{1+4\text{i}}$ **[6]**
   b) **Hence** solve $\dfrac{3+2\text{i}}{z} = 10-2\text{i}$, giving z in the form $a + b\text{i}$, where a and b are rational. **[2]**

5. On separate Argand diagrams, **sketch the locus** of z for each of the following:
   a) $|z-4\text{i}| = |z+9|$ b) $\arg(z-3\text{i}) = \dfrac{\pi}{4}$ **[4]**

6. **Shade the region** on an Argand diagram represented by $1 \le |z+6-3\text{i}| \le 3$ **[3]**

7. $f(z) = z^3 - 8z^2 + 30z - 36$
   a) **Verify that** $z = 3+3\text{i}$ is a solution to $f(z)=0$ **[4]**
   b) **Hence** show that $z^2 - 6z + 18$ is a factor of $f(z)$ **[2]**
   c) **Hence** solve $f(z)=0$ **completely** **[2]**

8. z is the complex number defined as $z = \dfrac{x-3\text{i}}{x+2\text{i}}$, where $x > 0$, $x \in \mathbb{R}$. The real part of z is $-\dfrac{1}{4}$.
   a) Find the value of x **[4]**
   b) **Hence** write z in the form $a + b\text{i}$, where a and b are rational numbers in their lowest terms **[1]**

9. **Shade the region** on an Argand diagram represented by
   $\{z \in \mathbb{C} : |z| \le 4\} \cap \left\{z \in \mathbb{C} : -\dfrac{\pi}{2} \le \arg(z-2) \le \dfrac{\pi}{2}\right\}$ **[3]**
   Hint: Consider the two conditions separately then find the region which satisfies both.

TOTAL 49 MARKS

Subtopics: Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots

1. Evaluate the following sums, showing your reasoning in each case:

   a) $\sum_{r=1}^{25}(2r-1)$ b) $\sum_{r=36}^{64}4r$

   c) $\sum_{r=1}^{15}(r^2-1)$ d) $\sum_{r=1}^{13}r^2(r+1)$ **[10]**

2. a) Find an expression for $\sum_{r=1}^{2n-1}(2r+1)$. Give your answer in the form an^2+bn+c. **[2]**

   b) Hence find the **smallest** integer N such that $\sum_{r=1}^{2N-1}(2r+1)>999$ **[2]**

3. Show that $\sum_{r=1}^{2n+1}(r-1)^2=\frac{1}{3}n(2n+1)(4n+1)$ **[3]**

4. The quartic equation $5x^4-9x^3+x^2+7x+3=0$ has roots α, β, γ and δ. **Without solving the equation**, find:

   a) $\alpha+\beta+\gamma+\delta$ **[2]**

   b) $\alpha\beta\gamma\delta$ **[2]**

   c) $\frac{1}{\alpha\beta\gamma}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\beta\gamma\delta}$ **[2]**

5. The cubic equation $2x^3+4x^2-3x-1=0$ has roots α, β and γ, where $\alpha+\beta+\gamma=-2$, $\alpha\beta+\alpha\gamma+\beta\gamma=-\frac{3}{2}$ and $\alpha\beta\gamma=\frac{1}{2}$

   The cubic equation $w^3+pw^2+5w+q=0$ has roots $2\alpha-1$, $2\beta-1$ and $2\gamma-1$. Find the integers p and q. **[4]**

6. The cubic equation $8x^3-36x^2+46x-15=0$ has roots α, $\alpha+1$ and $\alpha+2$. **Without solving the equation**, find the value of α. Show detailed reasoning for your answer. **[3]**

7. a) Show that $\sum_{r=1}^{n^2}(5r-6)=\frac{1}{2}n^2(5n^2-7)$ **[2]**

   b) Hence show that there is **no positive integer** n that satisfies $\sum_{r=1}^{n^2}(5r-6)=\sum_{r=1}^{n}36r^3$ **[3]**

8. The roots of the cubic equation $ax^3+bx^2+cx+d=0$ are $\alpha=\frac{1}{3}$, $\beta=-\frac{1}{3}$ and $\gamma=9$. Find **integer** values for a, b, c and d. **[5]**

TOTAL 40 MARKS

Volumes of Revolution – Test A (60 mins)

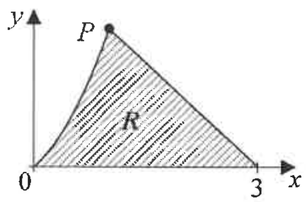
Fundamentals ✓Challenge Expert

Subtopics: Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes

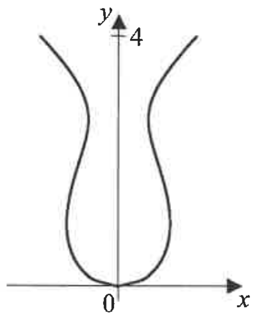
1. Use **calculus** to find the **exact** volumes of the solids generated when the following curves are rotated through 360° about the ***x*-axis** between the given limits:
   a) $y = 7x^3$ between $x = 0$ and $x = 1$ **[3]**
   b) $y = 5x^2 + 1$ between $x = -1$ and $x = 2$ **[3]**
   c) $y^2 - x^2 = 1$ between $x = -2$ and $x = 1$ **[3]**

2. Use **calculus** to find the volumes of the solids generated when the following curves are rotated through 2π radians about the ***y*-axis** between the given limits. Give your answers to **3 significant figures**.
   a) $x = \sqrt{4y^3 - 5y^2 + 5}$ between $y = 0$ and $y = 1$ **[3]**
   b) $x = 6y^{\frac{5}{2}} + 2y^{\frac{1}{2}}$ between $y = 1$ and $y = \sqrt{2}$ **[3]**
   c) $x^2y^2 = y^2 + 1$ between $y = 1$ and $y = 2$ **[4]**

3. The line with equation $y = 3 - x$ and the curve with equation $y = x^2 + x$ intersect at the point P. The finite region bounded by the x-axis, the curve and the line is R, as illustrated to the right.
   a) Find the **coordinates** of the point P shown in the diagram. **[2]**
   b) Calculate the volume of revolution when R is rotated through 360° about the ***x*-axis**. Give your answer in the form $k\pi$, where k is a **rational** number. **[6]**



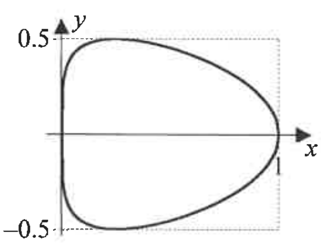
4. The cross section of a drop of water hanging from a ceiling is modelled by the equation $2y^3 - 11y^2 + 16y - 8x^2 = 0$ for $0 \le y \le 4$, as sketched to the right, where each unit on the axes represents 1 mm.
   By rotating the curve about a suitable axis, use this model to estimate the volume of the drop of water to **3 significant figures**. **[4]**



5. The curve $x = \lambda y^{\frac{1}{2}} + 6y^{\frac{3}{2}}$, where $\lambda > 0$ is a constant, is rotated through 360° about the ***y*-axis** between $y = 1$ and $y = 2$. This generates a solid of revolution with volume 175π. Find the value of λ. **[6]**

6. The finite region R is bounded by the lines $x = r$ and $y = h$, where $r, h > 0$, and the two coordinate axes. When R is rotated through 2π radians about the ***y*-axis**, it generates a cylinder of radius r and height h. Use **integration** to prove that the volume V of this cylinder is given by $V = \pi r^2 h$. **[3]**

7. An artist plans to carve a sculpture from a block of ice. The shape of the sculpture is modelled by rotating the curve $y^2 + x = \sqrt{x}$ (sketched to the right) about the ***x*-axis** for $0 \le x \le 1$, where each unit on the axes corresponds to 1 m.



   a) Use integration to estimate the volume of the sculpture to 3 significant figures. **[3]**

   The artist plans to carve the sculpture from a **cylindrical** block of ice of height 1 m and radius 0.5 m.
   b) Estimate how much ice the sculptor carves away from the cylinder. **[2]**
   c) Suggest a limitation of using this model to estimate these volumes. **[1]**

TOTAL 46 MARKS

Subtopics: Matrices, matrix multiplication, determinants, inverting 2 × 2 and 3 × 3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation

1. For the matrices $\mathbf{A} = \begin{pmatrix} 3 & 3 \\ -2 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & -2 \\ -3 & 0 \end{pmatrix}$, **without** using a calculator, find:

   a) $\mathbf{B} - \mathbf{A}$ b) $3\mathbf{A} - 2\mathbf{B}$ c) $\mathbf{AB}$ d) $\mathbf{A}^2$ e) $\mathbf{B}^{-1}$ f) $\mathbf{A}^{\mathrm{T}}$ **[11]**

2. Find matrix representations of the linear transformations $P: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - y \\ -4y \end{pmatrix}$ and $Q: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3y \\ 2x \end{pmatrix}$ **[2]**

3. a) $\mathbf{M} = \begin{pmatrix} 4 & -1 & 0 \\ 1 & 2 & 3 \\ -2 & -2 & 1 \end{pmatrix}$. **Without** using a calculator, show that **M** is **non-singular**. **[3]**

   b) Hence find $\mathbf{M}^{-1}$ **without** using a calculator. **[4]**

4. Triangle T with vertices $A(1, 1)$, $B(1, 6)$ and $C(5, 1)$ maps to triangle T' under the matrix $\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$

   a) Find the coordinates of the vertices of T' **[3]**
   b) Find the area of T' **[2]**

5. a) **Describe fully** the transformations represented by matrices $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ **[4]**

   b) Find the coordinates of the image of the point (3, –4) after being transformed by matrix **A** **[1]**
   c) State the coordinates of the **invariant point** of the transformation represented by matrix **B** **[1]**

6. The transformation R is an anticlockwise rotation about the origin through an angle $0° < \theta < 360°$, represented by the 2 × 2 matrix **R**. The transformation E is an enlargement, scale factor $k > 0$, centred at the origin, represented by the 2 × 2 matrix **E**.

   a) Write down the determinant of **R** and the determinant of **E** in terms of k **[3]**

   The matrix $\mathbf{M} = \begin{pmatrix} -3\sqrt{3} & -3 \\ 3 & -3\sqrt{3} \end{pmatrix}$ represents the combined transformation of R followed by E

   b) Find the value of k and the value of θ **[5]**

   Given that **M** maps point P with coordinates (x, y) onto point P' with coordinates (a, b),

   c) find the coordinates of P in terms of a and b **[4]**

7. $\mathbf{A} = \begin{pmatrix} -2 & a & 1 \\ 1 & 4 & c \\ 3 & b & 2 \end{pmatrix}$. Given that $\mathbf{A}^2 = \begin{pmatrix} 10 & 4 & 12 \\ 14 & 11 & 25 \\ -2 & -3 & -1 \end{pmatrix}$, find the values of a, b and c. **[3]**

8. Find the 3 × 3 matrices representing the following transformations:

   a) A reflection in the plane $y = 0$ **[2]**
   b) A rotation of 90° about the x-axis **[2]**

9. Three planes A, B and C are defined by the equations:

   A: $6x + 3y + 2z = 10$

   B: $3x + y + 2z = 4$

   C: $-2y + az = -4$, where a is a constant.

   a) Given that the planes **do not** meet at a single point, find the value of the constant a **[4]**
   b) Determine whether the system is **consistent**, and interpret it geometrically **[4]**

TOTAL 58 MARKS

Subtopics: Proof by mathematical induction, proving divisibility results, proving statements involving matrices

1. Prove by induction that $\sum_{r=1}^{n}(2r-1)=n^2$ for all positive integers n **[5]**

2. Prove by induction that $\sum_{r=1}^{n}(r^2+3r)=\frac{1}{3}n(n+1)(n+5)$ for all positive integers n **[6]**

3. Let $f(n)=7^n+5$. Prove by induction that $f(n)$ is divisible by 6 for all positive integers n. **[6]**

4. Let $\mathbf{A}=\begin{pmatrix}5 & 4\\ -4 & -3\end{pmatrix}$. Prove by induction that $\mathbf{A}^n=\begin{pmatrix}1+4n & 4n\\ -4n & 1-4n\end{pmatrix}$ for every positive integer n. **[6]**

5. Prove by induction that $\sum_{r=1}^{n}2^{r-1}=2^n-1$ for all positive integers n **[5]**

6. Let $f(n)=n^2$. Ethel tries to prove that $f(n)$ is odd for every natural number n. Her attempt is shown below.

> When $n=1$, $f(1)=1^2=1$, which is odd
> Assume the statement is true when $n=k$, so $f(k)$ is odd
> Then $f(k+1)-f(k)=(k+1)^2-k^2$
> $=k^2+2k+1-k^2$
> $=2k+1$
> So $f(k+1)=f(k)+2k+1$
> Since $f(k)$ and $2k+1$ are both odd, $f(k+1)$ is also odd
> So if the statement is true for $n=k$, then it is true for $n=k+1$. Since it is true for $n=1$, by the principle of mathematical induction it is true for all natural numbers n.

   a) Identify the error in Ethel's working and state in which step of the proof by induction this error occurs. **[2]**
   b) Find a counterexample to the statement that Ethel is trying to prove. **[1]**

7. Let $f(n)=4^n-3n+2$
   a) Calculate $f(1)$, $f(2)$, $f(3)$, and $f(4)$ **[2]**
   b) Based on your answer to part a), suggest an integer d in the range $1<d<10$ that has the property that d divides $f(n)$ for every positive integer n **[1]**
   c) Prove your conjecture from part b) using induction **[6]**

8. Let $\mathbf{B}=\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}$. Prove by induction that $\mathbf{B}^n=\begin{pmatrix}2^n & 2^n-1\\ 0 & 1\end{pmatrix}$ for every natural number n. **[6]**

TOTAL 46 MARKS

Subtopics: Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars

1. The line L passes through the point (3, 2, –1) and is **parallel** to the vector $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
   a) Write down a **vector equation** for L in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ **[1]**
   b) Write down a **Cartesian equation** for L in the form $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ **[1]**

2. Let $\mathbf{a} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$
   a) Find **a.b** **[2]**
   b) Find the acute angle between **a** and **b** correct to the **nearest degree** **[3]**

3. The line l_1 has vector equation $\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The line l_2 has Cartesian equation $x+1 = y-1 = z+1$.
   a) Write a vector equation for l_2 in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{b}$ **[1]**
   b) Find the **exact shortest distance** between the lines l_1 and l_2 **[6]**

4. a) Find the coordinates of the **point of intersection** of the line given by the vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ with the plane given by the Cartesian equation $x + 3y - z = -10$ **[3]**
   b) Find the **acute** angle between this line and this plane correct to **3 significant figures** **[3]**

5. Three points are given by $A(-2, 3, 7)$, $B(-1, 2, 7)$ and $C(1, 6, 6)$
   a) Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the plane that passes through A, B and C **[3]**
   b) The point D has coordinates (4, 4, 0). Show that A, B, C and D are **not** coplanar. **[3]**

6. a) Find a vector equation for the line l_1 that passes through the points (5, 6, –2) and (3, 3, 1) **[2]**
   The line l_2 has vector equation $\mathbf{r} = (3\mathbf{i} - 2\mathbf{k}) + \mu(2\mathbf{j} + 3\mathbf{k})$
   b) Show that the lines l_1 and l_2 are **skew** **[4]**

7. Let $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = -5\mathbf{j} + p\mathbf{k}$ for some constant p. The vectors **a** and **b** are **perpendicular**. Find the value of p. **[3]**

8. The line l_1 has vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ -3 \\ 19 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$. The line l_2 has vector equation $\mathbf{r} = \begin{pmatrix} 16 \\ -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -8 \\ 6 \\ -2 \end{pmatrix}$.
   Verify that the lines l_1 and l_2 intersect, and find the coordinates of their point of intersection. **[5]**

9. The line L has vector equation $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}) + \lambda(-\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$. The point P has coordinates (–1, –5, 4). Find the coordinates of the **reflection** of the point P in the line L. **[6]**

10. Relative to a fixed origin, O, an airport's air traffic control tower is modelled as a point with position vector $(9\mathbf{i} - \mathbf{j})$ km. The airport is responsible for any aeroplane that flies within 3 km of the tower. An aeroplane flies in a straight line from the point A with position vector $(-15\mathbf{i} + 7\mathbf{j})$ km to the point B with position vector $(19\mathbf{i} - 9\mathbf{j} + 2\mathbf{k})$ km. Determine whether the airport is responsible for the aeroplane during this flight. **[8]**

TOTAL 54 MARKS

1. $z_1 = -2+5\text{i},\ z_2 = 3-4\text{i}$

a) $z_2 - z_1 = (3-4\text{i})-(-2+5\text{i}) = 5-9\text{i}$ **A1**

b) $2z_1 + 4z_2 = 2(-2+5\text{i})+4(3-4\text{i}) = -4+10\text{i}+12-16\text{i}$ **M1**

$= 8-6\text{i}$ **A1**

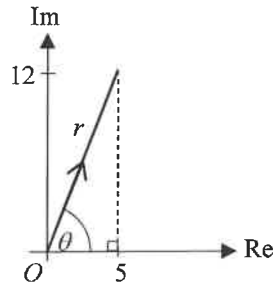
c) $z_1 z_2 = (-2+5\text{i})(3-4\text{i}) = -6+8\text{i}+15\text{i}-20\text{i}^2$ **M1**

$= -6-20(-1)+23\text{i} = 14+23\text{i}$ **A1**

d) $z_2^{\ *} = 3+4\text{i}$ **A1** **[6 Marks]**

> **Hint:** z^* is the complex conjugate of $z = a + bi$, and is equal to $a - bi$

2. a) $z = 5+12\text{i}$

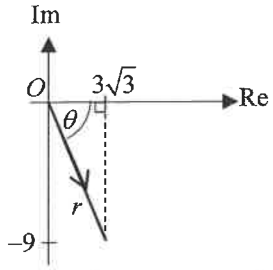


$r = \sqrt{5^2+12^2} = \sqrt{169} = 13$ **M1**

$\theta = \arg z = \arctan\left(\frac{12}{5}\right) = 1.17600... = 1.18$ (3 s.f.) **M1**

$\therefore 5+12\text{i} = 13(\cos 1.18 + \text{i}\sin 1.18)$ **A1**

b) $z = 3\sqrt{3}-9\text{i}$



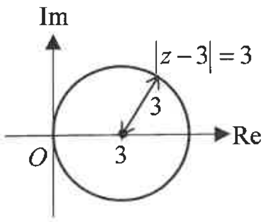
$r = \sqrt{\left(3\sqrt{3}\right)^2 + (-9)^2} = \sqrt{108} = 6\sqrt{3}$ **M1**

$\theta = \arg z = -\arctan\left(\frac{9}{3\sqrt{3}}\right) = -\frac{\pi}{3}$ **M1**

$\therefore 3\sqrt{3}-9\text{i} = 6\sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right)+\text{i}\sin\left(-\frac{\pi}{3}\right)\right)$ **A1** **[6 Marks]**

> **Tip:** Sketch an Argand diagram and consider the lengths in a right-angled triangle, i.e. use positive values in the arctan calculation, and then use your diagram to help you work out the argument so that $-\pi < \theta \leq \pi$

3. a)



A1 circle with centre (3, 0)
A1 radius 3

Let $z = x + \text{i}y$

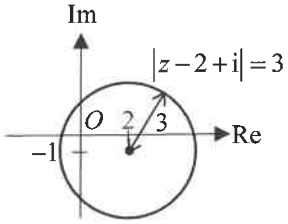
Then $|x + \text{i}y - 3| = 3$

So $\sqrt{(x-3)^2 + y^2} = 3$

So Cartesian equation is $(x-3)^2 + y^2 = 9$ **A1**

> **Alternatively:** Recognise that this is a circle with centre (3, 0) and radius 3, and state the Cartesian equation of a circle

b)



A1 circle with centre (2, −1)
A1 radius 3

Let $z = x + \mathrm{i}y$
Then $|x + \mathrm{i}y - 2 + \mathrm{i}| = 3$
So $\sqrt{(x-2)^2 + (y+1)^2} = 3$

So Cartesian equation is $(x-2)^2 + (y+1)^2 = 9$ **A1** **[6 Marks]**

> **Technique:** Multiply the numerator and the denominator by the complex conjugate of the denominator to remove the imaginary component from the denominator

4. a) i) $\dfrac{3+2\mathrm{i}}{5-\mathrm{i}} = \dfrac{3+2\mathrm{i}}{5-\mathrm{i}} \times \dfrac{5+\mathrm{i}}{5+\mathrm{i}} = \dfrac{15+3\mathrm{i}+10\mathrm{i}+2\mathrm{i}^2}{25+5\mathrm{i}-5\mathrm{i}-\mathrm{i}^2}$ **M1**

$= \dfrac{15+13\mathrm{i}-2}{25-(-1)} = \dfrac{13+13\mathrm{i}}{26} = \dfrac{1}{2} + \dfrac{1}{2}\mathrm{i}$ **A1**

ii) $\dfrac{6+\mathrm{i}}{2+\mathrm{i}} = \dfrac{6+\mathrm{i}}{2+\mathrm{i}} \times \dfrac{2-\mathrm{i}}{2-\mathrm{i}} = \dfrac{12-6\mathrm{i}+2\mathrm{i}-\mathrm{i}^2}{4-2\mathrm{i}+2\mathrm{i}-\mathrm{i}^2}$ **M1**

$= \dfrac{12-4\mathrm{i}-(-1)}{4-(-1)} = \dfrac{13-4\mathrm{i}}{5} = \dfrac{13}{5} - \dfrac{4}{5}\mathrm{i}$ **A1**

iii) $\dfrac{2-\mathrm{i}}{1+4\mathrm{i}} = \dfrac{2-\mathrm{i}}{1+4\mathrm{i}} \times \dfrac{1-4\mathrm{i}}{1-4\mathrm{i}} = \dfrac{2-8\mathrm{i}-\mathrm{i}+4\mathrm{i}^2}{1-4\mathrm{i}+4\mathrm{i}-16\mathrm{i}^2}$ **M1**

$= \dfrac{2-9\mathrm{i}-4}{1-(-16)} = \dfrac{-2-9\mathrm{i}}{17} = -\dfrac{2}{17} - \dfrac{9}{17}\mathrm{i}$ **A1**

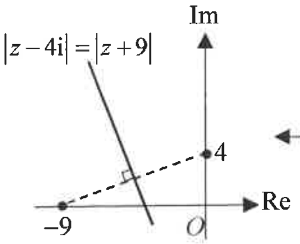
b) $\dfrac{3+2\mathrm{i}}{z} = 10-2\mathrm{i} \therefore \dfrac{3+2\mathrm{i}}{10-2\mathrm{i}} = \dfrac{3+2\mathrm{i}}{2(5-\mathrm{i})} = z$

> **Technique:** Recognise that the numerator is the same as in part ai), and the right-hand side is a multiple of the denominator from part ai)

So $\dfrac{3+2\mathrm{i}}{5-\mathrm{i}} = 2z$ **M1**

From part ai), $2z = \dfrac{1}{2} + \dfrac{1}{2}\mathrm{i} \therefore z = \dfrac{1}{4} + \dfrac{1}{4}\mathrm{i}$ **A1** **[8 Marks]**

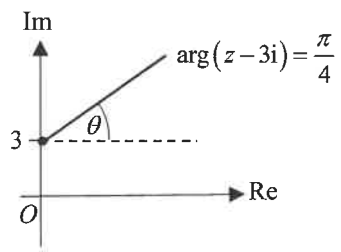
5. a) $|z-4\mathrm{i}| = |z+9|$



A1 perpendicular bisector
A1 (−9, 0) and (0, 4) shown

> **Technique:** The locus of $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line joining the points representing z_1 and z_2

b)

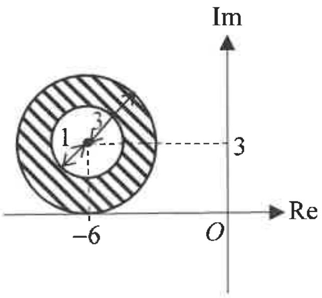


A1 half-line from (0, 3)

A1 at an angle of $\theta = \dfrac{\pi}{4}$ **[4 Marks]**

> **Technique:** The locus of $\arg(z - x - \mathrm{i}y) = \theta$ is a half-line from (x, y) making an angle of θ with a horizontal line passing through (x, y)

6.



A1 circle with centre (−6, 3) and radius 1
A1 circle with centre (−6, 3) and radius 3
A1 area between two circles shaded

> **Hint:** The locus of $|z - a - b\mathrm{i}| = r$ is a circle of radius r and centre (a, b)

[3 Marks]

7. $f(z) = z^3 - 8z^2 + 30z - 36$

a) **Verify that $z = 3 + 3i$ is a solution to $f(z) = 0$**

$(3+3i)^2 = 9 + 9i + 9i + 9i^2 = 9 - 9 + 18i = 18i$ **M1**

$(3+3i)^3 = (3+3i)^2(3+3i) = 18i(3+3i) = 54i + 54i^2 = -54 + 54i$ **M1**

So $f(3+3i) = (-54+54i) - 8\times(18i) + 30\times(3+3i) - 36$ **M1**

$= -54 + 54i - 144i + 90 - 90i - 36 = 0$ **A1**

b) **Show that $z^2 - 6z + 18$ is a factor of $f(z)$**

3 + 3i is a solution to $f(z) = 0$, so 3 – 3i is also a solution (since complex roots occur in conjugate pairs)

Therefore, $(z - 3 - 3i)$ and $(z - 3 + 3i)$ are both factors of $f(z)$ **M1**

So $(z - 3 - 3i)(z - 3 + 3i) = z^2 - 6z + 9 - 9i^2 = z^2 - 6z + 18$ is a factor of $f(z)$ **A1**

c)

$$\begin{array}{r} z-2 \\ z^2-6z+18\overline{\big)\,z^3-8z^2+30z-36} \\ -\left(z^3-6z^2+18z\right) \\ \hline -2z^2+12z-36 \\ -\left(-2z^2+12z-36\right) \\ \hline 0 \end{array}$$

Technique: $z^2 - 6z + 18$ is a factor of f(z) so f(z) can be written as $(z^2 - 6z + 18)(az + b)$. Use long division or inspection to find the linear factor.

So $f(z) = (z^2 - 6z + 18)(z - 2)$ **M1**

So the roots of $f(z) = 0$ are 3 + 3i, 3 – 3i, 2 **A1** **[8 Marks]**

8. $z = \dfrac{x - 3i}{x + 2i}$

Technique: Write z in the form $a + bi$ by multiplying both the numerator and the denominator by the complex conjugate of the denominator. Then set a equal to $-\frac{1}{4}$.

a) $\dfrac{x-3i}{x+2i} \times \dfrac{x-2i}{x-2i} = \dfrac{x^2 - 2xi - 3xi + 6i^2}{x^2 - 2xi + 2xi - 4i^2} = \dfrac{x^2 - 6 - 5xi}{x^2 + 4}$ **M1**

The real part is $\dfrac{x^2-6}{x^2+4}$ so $\dfrac{x^2-6}{x^2+4} = -\dfrac{1}{4}$ **M1**

So $x^2 - 6 = -\dfrac{x^2}{4} - 1$

$\therefore \dfrac{5}{4}x^2 = 5$

$x^2 = 4 \therefore x = \pm 2$ **M1**

But $x > 0$ so $x = 2$ **A1**

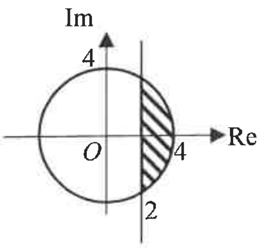
b) Let $z = a + bi$

We know that the real part is $-\dfrac{1}{4}$ so $a = -\dfrac{1}{4}$

The imaginary part is $b = \dfrac{-5x}{x^2+4} = \dfrac{-5\times 2}{2^2+4} = -\dfrac{10}{8} = -\dfrac{5}{4}$

So $z = -\dfrac{1}{4} - \dfrac{5}{4}i$ **A1** **[5 Marks]**

9. $\{z \in \mathbb{C} : |z| \le 4\} \cap \left\{z \in \mathbb{C} : -\dfrac{\pi}{2} \le \arg(z-2) \le \dfrac{\pi}{2}\right\}$



A1 circle with centre (0, 0) and radius 4

A1 vertical line at $x = 2$

A1 shaded region inside of circle and to the right of the vertical line **[3 Marks]**

Technique: The $\arg(z - 2)$ inequality produces a total angle of π centred on (2, 0)

TOTAL 49 MARKS

1. a) $\sum_{r=1}^{25}(2r-1)=2\sum_{r=1}^{25}r-\sum_{r=1}^{25}1$

$=2\times\frac{1}{2}\times25\times(25+1)-25$ **M1**

$=625$ **A1**

> **Technique:** $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$

b) $\sum_{r=36}^{64}4r=\sum_{r=1}^{64}4r-\sum_{r=1}^{35}4r$ **M1**

$=4\sum_{r=1}^{64}r-4\sum_{r=1}^{35}r$

$=4\times\frac{1}{2}\times64\times(64+1)-4\times\frac{1}{2}\times35\times(35+1)$ **M1**

$=5800$ **A1**

> **Tip:** Most modern calculators will calculate these kinds of sum. You can use a calculator to check your answer, but since the question asks you to show your reasoning you should show your calculations using the formulae.

c) $\sum_{r=1}^{15}(r^2-1)=\sum_{r=1}^{15}r^2-\sum_{r=1}^{15}1$

$=\frac{1}{6}\times15\times(15+1)\times(2\times15+1)-15$ **M1**

$=1225$ **A1**

> **Technique:** $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$

d) $\sum_{r=1}^{13}r^2(r+1)=\sum_{r=1}^{13}(r^3+r^2)$ **M1**

$=\sum_{r=1}^{13}r^3+\sum_{r=1}^{13}r^2$

$=\frac{1}{4}\times13^2\times(13+1)^2+\frac{1}{6}\times13\times(13+1)\times(2\times13+1)$ **M1**

$=9100$ **A1** **[10 Marks]**

> **Technique:** $\sum_{r=1}^{n}r^3=\frac{1}{4}n^2(n+1)^2$

2. a) $\sum_{r=1}^{2n-1}(2r+1)=2\sum_{r=1}^{2n-1}r+\sum_{r=1}^{2n-1}1$

$=2\times\frac{1}{2}(2n-1)((2n-1)+1)+(2n-1)$ **M1**

$=(2n-1)(2n)+2n-1$

$=4n^2-1$ **A1**

> **Technique:** Use the usual formula for the sum of the first *n* natural numbers, but replace every instance of *n* with (2*n* – 1)

b) We want the smallest integer N such that $\sum_{r=1}^{2N-1}(2r+1)>999$

By part a) this is the smallest integer N such that $4N^2-1>999$ **M1**

$\therefore 4N^2>1000$

$\therefore N^2>\frac{1000}{4}=250$

$\therefore N>\sqrt{250}=15.8113...$ or $N<-\sqrt{250}=-15.8113...$

But N needs to be positive, so we can ignore $N<-\sqrt{250}$

So the smallest integer value of N is 16 **A1** **[4 Marks]**

3. **Show that** $\sum_{r=1}^{2n+1}(r-1)^2 = \frac{1}{3}n(2n+1)(4n+1)$

> **Alternatively:** Notice that $\sum_{r=1}^{2n+1}(r-1)^2 = \sum_{r=0}^{2n} r^2 = \sum_{r=1}^{2n} r^2$ and use the usual formula for the sum of squares

$$\sum_{r=1}^{2n+1}(r-1)^2 = \sum_{r=1}^{2n+1}(r^2-2r+1) \quad \textbf{M1}$$

$$= \sum_{r=1}^{2n+1} r^2 - 2\sum_{r=1}^{2n+1} r + \sum_{r=1}^{2n+1} 1$$

$$= \frac{1}{6}(2n+1)(2n+1+1)(2\times(2n+1)+1) - 2\times\frac{1}{2}(2n+1)(2n+1+1)+(2n+1) \quad \textbf{M1}$$

$$= \frac{1}{6}(2n+1)\left[(2n+2)(4n+3)-6(2n+2)+6\right]$$

$$= \frac{1}{6}(2n+1)(8n^2+2n)$$

$$= \frac{1}{3}n(2n+1)(4n+1) \quad \textbf{A1}$$

[3 Marks]

4. $5x^4 - 9x^3 + x^2 + 7x + 3 = 0$ has roots α, β, γ and δ

a) $$\alpha+\beta+\gamma+\delta = -\frac{b}{a}$$

$$= -\frac{-9}{5} \quad \textbf{M1}$$

$$= \frac{9}{5} \quad \textbf{A1}$$

b) $$\alpha\beta\gamma\delta = \frac{e}{a}$$

$$= \frac{3}{5} \quad \textbf{M1A1}$$

c) $$\frac{1}{\alpha\beta\gamma}+\frac{1}{\alpha\beta\delta}+\frac{1}{\alpha\gamma\delta}+\frac{1}{\beta\gamma\delta} = \frac{\alpha+\beta+\gamma+\delta}{\alpha\beta\gamma\delta}$$

$$= \frac{9/5}{3/5} \quad \textbf{M1}$$

$$= 3 \quad \textbf{A1}$$

[6 Marks]

5. $2x^3 + 4x^2 - 3x - 1 = 0$ has roots α, β and γ, where $\alpha+\beta+\gamma = -2$, $\alpha\beta+\alpha\gamma+\beta\gamma = -\frac{3}{2}$ and $\alpha\beta\gamma = \frac{1}{2}$

$w^3 + pw^2 + 5w + q = 0$ has roots $2\alpha - 1, 2\beta - 1$ and $2\gamma - 1$

> **Alternatively:** Let $w = 2x - 1$, so that $x = \frac{w+1}{2}$. Then substitute this expression for x into the original equation $2x^3 + 4x^2 - 3x - 1 = 0$ to get an equation with roots $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$. After multiplying by a suitable integer you will have an equation of the form $w^3 + pw^2 + 5w + q = 0$.

So $-p = (2\alpha-1)+(2\beta-1)+(2\gamma-1)$

$$= 2(\alpha+\beta+\gamma)-3 \quad \textbf{M1}$$

$$= 2\times(-2)-3$$

$$= -7$$

So $p = 7$ **A1**

And $-q = (2\alpha-1)(2\beta-1)(2\gamma-1)$

$$= (2\alpha-1)(4\beta\gamma-2\beta-2\gamma+1)$$

$$= 8\alpha\beta\gamma - 4\alpha\beta - 4\alpha\gamma + 2\alpha - 4\beta\gamma + 2\beta + 2\gamma - 1$$

$$= 8(\alpha\beta\gamma) - 4(\alpha\beta+\alpha\gamma+\beta\gamma) + 2(\alpha+\beta+\gamma) - 1 \quad \textbf{M1}$$

$$= 8\times\frac{1}{2} - 4\times\left(-\frac{3}{2}\right) + 2\times(-2) - 1$$

$$= 5$$

So $q = -5$ **A1**

[4 Marks]

6. $8x^3 - 36x^2 + 46x - 15 = 0$ has roots α, $\alpha + 1$ and $\alpha + 2$

$$\alpha + \alpha + 1 + \alpha + 2 = -\frac{b}{a}$$

Tip: You could use any of the formulae relating the roots of a polynomial to its coefficients, but this is the only one that gives you a linear equation for α

$$= -\frac{-36}{8} \quad \textbf{M1}$$

$$= \frac{9}{2}$$

So, $3\alpha + 3 = \frac{9}{2}$ **M1**

And so $\alpha = \frac{9/2 - 3}{3} = \frac{1}{2}$ **A1** **[3 Marks]**

7. a) **Show that $\sum_{r=1}^{n^2} (5r-6) = \frac{1}{2}n^2(5n^2 - 7)$**

$$\sum_{r=1}^{n^2} (5r-6) = 5\sum_{r=1}^{n^2} r - 6\sum_{r=1}^{n^2} 1$$

$$= 5 \times \frac{1}{2}n^2(n^2+1) - 6n^2 \quad \textbf{M1}$$

$$= \frac{1}{2}n^2(5(n^2+1) - 12)$$

$$= \frac{1}{2}n^2(5n^2 - 7) \quad \textbf{A1}$$

b) **Show that there is no positive integer n that satisfies $\sum_{r=1}^{n^2} (5r-6) = \sum_{r=1}^{n} 36r^3$**

We will prove the statement by contradiction

Assume that the positive integer n satisfies $\sum_{r=1}^{n^2} (5r-6) = \sum_{r=1}^{n} 36r^3$

So $\frac{1}{2}n^2(5n^2 - 7) = 9n^2(n+1)^2$ **M1**

$\therefore (5n^2 - 7) = 18(n+1)^2$

$\therefore 5n^2 - 7 = 18n^2 + 36n + 18$

$\therefore 13n^2 + 36n + 25 = 0$

This quadratic equation has discriminant $b^2 - 4ac = 36^2 - 4 \times 13 \times 25 = -4 < 0$ **M1**

So there are no real values of n that satisfy the equation

This contradicts the assumption that n is an integer that satisfies the equation, and so there are no positive integers n such that $\sum_{r=1}^{n^2} (5r-6) = \sum_{r=1}^{n} 36r^3$ **A1** **[5 Marks]**

8. $ax^3 + bx^2 + cx + d = 0$ has roots $\alpha = \frac{1}{3}$, $\beta = -\frac{1}{3}$ and $\gamma = 9$

$-\frac{b}{a} = \alpha + \beta + \gamma = \frac{1}{3} - \frac{1}{3} + 9 = 9$, so $\frac{b}{a} = -9$ **A1**

Alternatively: You can also expand $\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)(x - 9)$ then multiply the answer by a suitable integer to make a polynomial with integer coefficients

$\frac{c}{a} = \alpha\beta + \alpha\gamma + \beta\gamma = \frac{1}{3} \times \left(-\frac{1}{3}\right) + \frac{1}{3} \times 9 + \left(-\frac{1}{3}\right) \times 9 = -\frac{1}{9}$ **A1**

$-\frac{d}{a} = \alpha\beta\gamma = \frac{1}{3} \times \left(-\frac{1}{3}\right) \times 9 = -1$, so $\frac{d}{a} = 1$ **A1**

The cubic equation may be written $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Substituting in the above values, this is:

$x^3 - 9x^2 - \frac{1}{9}x + 1 = 0$ **A1**

Multiply through by 9 to get integer coefficients:

$9x^3 - 81x^2 - x + 9 = 0$

So that $a = 9$, $b = -81$, $c = -1$ and $d = 9$ **A1**
[Allow any integer multiple of these values] **[5 Marks]**

TOTAL 40 MARKS

1. a) The curve has equation $y = 7x^3$

$\therefore V = \pi\int_0^1 (7x^3)^2 \, dx$ **M1**

$= \pi\int_0^1 49x^6 \, dx$

$= \pi\left[7x^7\right]_0^1$ **A1**

$= \pi(7\times1 - 7\times0)$

$= 7\pi$ **A1**

> **Technique:** If the curve with equation $y = f(x)$ is rotated 360° about the x-axis between $x = a$ and $x = b$, then the volume of the solid generated is given by $V = \pi\int_a^b y^2 \, dx = \pi\int_a^b (f(x))^2 \, dx$

b) The curve has equation $y = 5x^2 + 1$

$\therefore V = \pi\int_{-1}^2 (5x^2+1)^2 \, dx$ **M1**

$= \pi\int_{-1}^2 (25x^4 + 10x^2 + 1) \, dx$

$= \pi\left[5x^5 + \frac{10}{3}x^3 + x\right]_{-1}^2$ **A1**

$= \pi\left(5\times32 + \frac{10}{3}\times8 + 2 - 5\times(-1) - \frac{10}{3}\times(-1) - (-1)\right)$

$= 198\pi$ **A1**

c) The curve has equation $y^2 - x^2 = 1$, so $y^2 = x^2 + 1$

$\therefore V = \pi\int_{-2}^1 (x^2+1) \, dx$ **M1**

$= \pi\left[\frac{1}{3}x^3 + x\right]_{-2}^1$ **A1**

$= \pi\left(\frac{1}{3}\times1 + 1 - \frac{1}{3}\times(-8) - (-2)\right)$

$= 6\pi$ **A1** **[9 Marks]**

2. a) The curve has equation $x = \sqrt{4y^3 - 5y^2 + 5}$

$\therefore V = \pi\int_0^1 \left(\sqrt{4y^3 - 5y^2 + 5}\right)^2 dy$ **M1**

$= \pi\int_0^1 (4y^3 - 5y^2 + 5) \, dy$

$= \pi\left[y^4 - \frac{5}{3}y^3 + 5y\right]_0^1$ **A1**

$= \pi\left(1 - \frac{5}{3}\times1 + 5\times1 - 0 + \frac{5}{3}\times0 - 5\times0\right)$

$= 13.6135... = 13.6$ (3 s.f.) **A1**

> **Technique:** If the curve with equation $x = f(y)$ is rotated 2π radians about the y-axis between $y = a$ and $y = b$, then the volume of the solid generated is given by $V = \pi\int_a^b x^2 \, dy = \pi\int_a^b (f(y))^2 \, dy$

b) The curve has equation $x = 6y^{\frac{5}{2}} + 2y^{\frac{1}{2}}$

$\therefore V = \pi\int_1^{\sqrt{2}} \left(6y^{\frac{5}{2}} + 2y^{\frac{1}{2}}\right)^2 dy$ **M1**

$= \pi\int_1^{\sqrt{2}} (36y^5 + 24y^3 + 4y) \, dy$

$= \pi\left[6y^6 + 6y^4 + 2y^2\right]_1^{\sqrt{2}}$ **A1**

$= \pi(6\times8 + 6\times4 + 2\times2 - 6\times1 - 6\times1 - 2\times1)$

$= 194.778... = 195$ (3 s.f.) **A1**

c) The curve has equation $x^2y^2 = y^2 + 1$

$\therefore x^2 = \frac{y^2+1}{y^2} = 1 + \frac{1}{y^2}$ **A1**

So $V = \pi \int_1^2 \left(1 + \frac{1}{y^2}\right) \mathrm{d}y$ **M1**

$= \pi \left[y - \frac{1}{y} \right]_1^2$ **A1**

$= \pi \left(2 - \frac{1}{2} - 1 + \frac{1}{1} \right)$

$= 4.71238... = 4.71$ (3 s.f.) **A1** **[10 Marks]**

3. a) P is an intersection of $y = 3 - x$ and $y = x^2 + x$, so its x-coordinate is a solution of $3 - x = x^2 + x$

$\therefore x^2 + 2x - 3 = 0$ **M1**

$\therefore (x+3)(x-1) = 0$

So $x = -3$ or $x = 1$

We can see from the diagram that $x > 0$, so $x = 1$. Hence $y = 3 - 1 = 2$.

So $P = (1, 2)$ **A1**

b) Split R into two regions, R_1 for $0 \le x \le 1$ and R_2 for $1 < x \le 3$. Let V_1 be the volume of the solid generated when R_1 is rotated 360° about the x-axis, and V_2 be the volume of the solid generated when R_2 is rotated 360° about the x-axis.

$V_1 = \pi \int_0^1 (x^2 + x)^2 \, \mathrm{d}x$ **M1**

$= \pi \int_0^1 (x^4 + 2x^3 + x^2) \mathrm{d}x$

$= \pi \left[\frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 \right]_0^1$

$= \pi \left(\frac{1}{5} \times 1 + \frac{1}{2} \times 1 + \frac{1}{3} \times 1 - \frac{1}{5} \times 0 - \frac{1}{2} \times 0 - \frac{1}{3} \times 0 \right)$

$= \frac{31}{30}\pi$ **A1**

$V_2 = \pi \int_1^3 (3 - x)^2 \, \mathrm{d}x$ **M1**

$= \pi \int_1^3 (9 - 6x + x^2) \mathrm{d}x$

$= \pi \left[9x - 3x^2 + \frac{1}{3}x^3 \right]_1^3$ **A1**

$= \pi \left(9 \times 3 - 3 \times 9 + \frac{1}{3} \times 27 - 9 \times 1 + 3 \times 1 - \frac{1}{3} \times 1 \right)$

$= \frac{8}{3}\pi$ **A1**

> **Alternatively:** The solid of revolution formed by R_2 is a cone with radius $r = 2$ and height $h = 2$, so its volume is $V_2 = \frac{1}{3}\pi r^2 h = \frac{8}{3}\pi$

The total volume, V, of the solid generated when the whole region R is rotated 360° about the x-axis is the sum of these two volumes:

$V = V_1 + V_2 = \frac{31}{30}\pi + \frac{8}{3}\pi = \frac{37}{10}\pi \left(\text{so } k = \frac{37}{10} \right)$ **A1** **[8 Marks]**

4. We need to rotate the curve about the y-axis to find the volume of the drop, so we need an expression for x^2

The curve has equation $2y^3 - 11y^2 + 16y - 8x^2 = 0$

$\therefore x^2 = \frac{1}{4}y^3 - \frac{11}{8}y^2 + 2y$ **A1**

So $V = \pi\int_0^4 \left(\frac{1}{4}y^3 - \frac{11}{8}y^2 + 2y\right) dy$ **M1**

$= \pi\left[\frac{1}{16}y^4 - \frac{11}{24}y^3 + y^2\right]_0^4$ **A1**

$= \pi\left(\frac{1}{16}\times 256 - \frac{11}{24}\times 64 + 16 - \frac{1}{16}\times 0 + \frac{11}{24}\times 0 - 0\right)$

$= 8.37758... = 8.38 \text{ mm}^3$ (3 s.f.) **A1** **[4 Marks]**

5. The curve has equation $x = \lambda y^{\frac{1}{2}} + 6y^{\frac{3}{2}}$

$\therefore V = \pi\int_1^2 \left(\lambda y^{\frac{1}{2}} + 6y^{\frac{3}{2}}\right)^2 dy$ **M1**

$= \pi\int_1^2 \left(\lambda^2 y + 12\lambda y^2 + 36y^3\right) dy$

$= \pi\left[\frac{\lambda^2}{2}y^2 + 4\lambda y^3 + 9y^4\right]_1^2$ **A1**

$= \pi\left(\frac{\lambda^2}{2}\times 4 + 4\lambda\times 8 + 9\times 16 - \frac{\lambda^2}{2}\times 1 - 4\lambda\times 1 - 9\times 1\right)$

$= \pi\left(\frac{3}{2}\lambda^2 + 28\lambda + 135\right)$ **A1**

We are told in the question that $V = 175\pi$, so $\frac{3}{2}\lambda^2 + 28\lambda + 135 = 175$ **M1**

$\therefore \frac{3}{2}\lambda^2 + 28\lambda - 40 = 0$

So $\lambda = \dfrac{-28 \pm \sqrt{28^2 - 4\times \frac{3}{2}\times(-40)}}{2\times\frac{3}{2}}$

$= \dfrac{-28\pm\sqrt{1024}}{3}$

$= \dfrac{-28\pm 32}{3}$

So $\lambda = -20$ or $\lambda = \frac{4}{3}$ **A1**

We are told in the question that $\lambda > 0$, so $\lambda = \frac{4}{3}$ **A1** **[6 Marks]**

6. **Prove that the volume of the cylinder is $V = \pi r^2 h$**

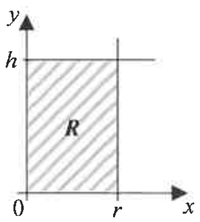
$V = \pi\int_0^h x^2\, dy$

$= \pi\int_0^h r^2\, dy$ **M1**

$= \pi\left[r^2 y\right]_0^h$ **A1**

$= \pi\left(r^2 h - r^2\times 0\right)$

$= \pi r^2 h$ **A1**



[3 Marks]

Tip: A sketch is not required by the question, but it may help you visualise the situation described

7. a) The curve has equation $y^2 + x = \sqrt{x}$, so $y^2 = \sqrt{x} - x$

So $V = \pi\int_0^1 \left(x^{\frac{1}{2}} - x\right)\mathrm{d}x$ **M1**

$= \pi\left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2\right]_0^1$ **A1**

$= \pi\left(\frac{2}{3}\times1 - \frac{1}{2}\times1 - \frac{2}{3}\times0 + \frac{1}{2}\times0\right)$

$= 0.523598... = 0.524\text{ m}^3$ (3 s.f.) **A1**

b) The volume of the cylinder is $\pi r^2 h = \pi\times0.5^2\times1 = \frac{1}{4}\pi$ **A1**

So the amount of ice carved away is $\frac{1}{4}\pi - 0.523598... = 0.261799... = 0.262\text{ m}^3$ (3 s.f.) **A1**

c) Any suitable problem with using the model **B1**
For example:

- Ice melts so the volume of the finished sculpture may be less than that given by the model
- Ice cannot be sculpted perfectly so the sculpture is unlikely to follow the curve precisely, and hence the volumes will be inaccurate

[6 Marks]

TOTAL 46 MARKS

1. $\mathbf{A}=\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & -2\\-3 & 0\end{pmatrix}$

a) $\mathbf{B}-\mathbf{A}=\begin{pmatrix}-1 & -2\\-3 & 0\end{pmatrix}-\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}=\begin{pmatrix}-4 & -5\\-1 & -4\end{pmatrix}$ **A1**

Technique: Subtract the corresponding elements in each matrix

b) $3\mathbf{A}-2\mathbf{B}=3\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}-2\begin{pmatrix}-1 & -2\\-3 & 0\end{pmatrix}=\begin{pmatrix}9 & 9\\-6 & 12\end{pmatrix}-\begin{pmatrix}-2 & -4\\-6 & 0\end{pmatrix}$ **M1**

$=\begin{pmatrix}11 & 13\\0 & 12\end{pmatrix}$ **A1**

Technique: Multiply each element in matrix **A** by 3 and each element in matrix **B** by 2, then subtract corresponding elements

c) Let $\mathbf{AB}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}\begin{pmatrix}-1 & -2\\-3 & 0\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$a=3\times(-1)+3\times(-3)=-12$

$b=3\times(-2)+3\times 0=-6$

$c=(-2)\times(-1)+4\times(-3)=-10$

$d=(-2)\times(-2)+4\times 0=4$ **M1**

So $\mathbf{AB}=\begin{pmatrix}-12 & -6\\-10 & 4\end{pmatrix}$ **A1**

Technique: To find each element in **AB**, find the sum of the elements in each row of **A** multiplied by the corresponding elements in each column of **B**

d) Let $\mathbf{A}^2=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

$\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}\begin{pmatrix}3 & 3\\-2 & 4\end{pmatrix}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$

Technique: To find $\mathbf{A}^2$, multiply **A** by itself

$a=3\times 3+3\times(-2)=3$

$b=3\times 3+3\times 4=21$

$c=(-2)\times 3+4\times(-2)=-14$

$d=(-2)\times 3+4\times 4=10$ **M1**

So $\mathbf{A}^2=\begin{pmatrix}3 & 21\\-14 & 10\end{pmatrix}$ **A1**

e) $\det\mathbf{B}=\begin{vmatrix}-1 & -2\\-3 & 0\end{vmatrix}=(-1)\times 0-(-2)\times(-3)$ **M1**

$=0-6=-6$ **A1**

So $\mathbf{B}^{-1}=\frac{1}{-6}\begin{pmatrix}0 & 2\\3 & -1\end{pmatrix}\left(\text{or}\begin{pmatrix}0 & -\frac{1}{3}\\-\frac{1}{2} & \frac{1}{6}\end{pmatrix}\right)$ **A1**

Technique: For a 2×2 matrix $\mathbf{M}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$, $\mathbf{M}^{-1}=\frac{1}{\det\mathbf{M}}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$, where $\det\mathbf{M}=ad-bc\ (\neq 0)$

Hint: $\mathbf{A}^{\mathrm{T}}$ means **A** is transposed, i.e. rows and columns are interchanged

f) $\mathbf{A}^{\mathrm{T}}=\begin{pmatrix}3 & -2\\3 & 4\end{pmatrix}$ **A1** **[11 Marks]**

2. The matrix representation of P: $\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}2x-y\\-4y\end{pmatrix}$ is $\begin{pmatrix}2 & -1\\0 & -4\end{pmatrix}$ **A1**

The matrix representation of Q: $\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}3y\\2x\end{pmatrix}$ is $\begin{pmatrix}0 & 3\\2 & 0\end{pmatrix}$ **A1** **[2 Marks]**

3. $\mathbf{M}=\begin{pmatrix}4 & -1 & 0\\ 1 & 2 & 3\\ -2 & -2 & 1\end{pmatrix}$

a) **Show that M is non-singular**

$$\det\mathbf{M}=\begin{vmatrix}4 & -1 & 0\\ 1 & 2 & 3\\ -2 & -2 & 1\end{vmatrix}=4\begin{vmatrix}2 & 3\\ -2 & 1\end{vmatrix}-(-1)\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix}+0\begin{vmatrix}1 & 2\\ -2 & -2\end{vmatrix}$$ **M1**

$$=4(2+6)+1(1+6)+0(-2+4)$$
$$=4\times8+1\times7+0\times2$$
$$=32+7+0=39$$ **A1**

Hence **M** is non-singular as $\det\mathbf{M}=39\neq0$ **A1**

Technique: The determinant of a 3×3 matrix $\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$ is $a\begin{vmatrix}e & f\\ h & i\end{vmatrix}-b\begin{vmatrix}d & f\\ g & i\end{vmatrix}+c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$

Hint: A matrix **M** is non-singular if $\det\mathbf{M}\neq0$, and singular if $\det\mathbf{M}=0$

b) To find $\mathbf{M}^{-1}$, first find the matrix of minors, **N**:

$$\mathbf{N}=\begin{pmatrix}\begin{vmatrix}2 & 3\\ -2 & 1\end{vmatrix} & \begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix} & \begin{vmatrix}1 & 2\\ -2 & -2\end{vmatrix}\\ \begin{vmatrix}-1 & 0\\ -2 & 1\end{vmatrix} & \begin{vmatrix}4 & 0\\ -2 & 1\end{vmatrix} & \begin{vmatrix}4 & -1\\ -2 & -2\end{vmatrix}\\ \begin{vmatrix}-1 & 0\\ 2 & 3\end{vmatrix} & \begin{vmatrix}4 & 0\\ 1 & 3\end{vmatrix} & \begin{vmatrix}4 & -1\\ 1 & 2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}8 & 7 & 2\\ -1 & 4 & -10\\ -3 & 12 & 9\end{pmatrix}$$ **M1A1**

Then find the transposed matrix of cofactors, $\mathbf{C}^{\mathrm{T}}$:

$$\mathbf{C}=\begin{pmatrix}8 & -7 & 2\\ 1 & 4 & 10\\ -3 & -12 & 9\end{pmatrix}\therefore\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}8 & 1 & -3\\ -7 & 4 & -12\\ 2 & 10 & 9\end{pmatrix}$$ **M1**

So $$\mathbf{M}^{-1}=\frac{1}{39}\begin{pmatrix}8 & 1 & -3\\ -7 & 4 & -12\\ 2 & 10 & 9\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}8/39 & 1/39 & -1/13\\ -7/39 & 4/39 & -4/13\\ 2/39 & 10/39 & 3/13\end{pmatrix}$$ **A1 [7 Marks]**

Tip: You can use your calculator to check that your inverse matrix is correct

4. The matrix is $\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}$

a) $A=(1, 1)$

$$\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}=\begin{pmatrix}2\times1+(-1)\times1\\ 2\times1+3\times1\end{pmatrix}=\begin{pmatrix}1\\ 5\end{pmatrix}$$ so $A'=(1, 5)$ **A1**

Tip: You could also combine this as a single matrix calculation by writing $\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1 & 5\\ 1 & 6 & 1\end{pmatrix}$

$B=(1, 6)$

$$\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 6\end{pmatrix}=\begin{pmatrix}2\times1+(-1)\times6\\ 2\times1+3\times6\end{pmatrix}=\begin{pmatrix}-4\\ 20\end{pmatrix}$$ so $B'=(-4, 20)$ **A1**

$C=(5, 1)$

$$\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}\begin{pmatrix}5\\ 1\end{pmatrix}=\begin{pmatrix}2\times5+(-1)\times1\\ 2\times5+3\times1\end{pmatrix}=\begin{pmatrix}9\\ 13\end{pmatrix}$$ so $C'=(9, 13)$ **A1**

Hint: Sketch a diagram showing points A, B and C

Hint: The determinant of a matrix representing a linear transformation represents the area scale factor for the transformation

b) Area of $T=\frac{1}{2}\times(5-1)\times(6-1)=\frac{1}{2}\times4\times5=10$ **M1**

The determinant of $\begin{pmatrix}2 & -1\\ 2 & 3\end{pmatrix}$ is $2\times3-(-1)\times2=8$ so the area of $T'=10\times8=80$ **A1** **[5 Marks]**

5. a) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents a reflection **A1**

in the line $y = x$ **A1**

> **Tip:** The first column represents the image of (1, 0) and the second column represents the image of (0, 1), hence this is a reflection in the line $y = x$

$\mathbf{B} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$ represents a rotation by angle θ anticlockwise about the origin **A1**

> **Tip:** The matrix for a rotation of θ anticlockwise about the origin is given in the formula book

$\cos\theta = \sin\theta = \frac{\sqrt{2}}{2}$ so $\theta = 45°$ $\left(\text{or } \frac{\pi}{4}\right)$ anticlockwise (or any equivalent description) **A1**

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ so the image of the point has coordinates (–4, 3) **A1**

c) **B** is a rotation about the origin, so the invariant point is the origin, (0, 0) **A1** **[6 Marks]**

6. a) R is an anticlockwise rotation about the origin through an angle $0° < \theta < 360°$ so $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

So $\det \mathbf{R} = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - (-\sin^2\theta)$ **M1**

$= \cos^2\theta + \sin^2\theta = 1$ **A1**

E is an enlargement, scale factor $k > 0$, centred at the origin, so $\mathbf{E} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

So $\det \mathbf{E} = \begin{vmatrix} k & 0 \\ 0 & k \end{vmatrix} = k^2 - 0 = k^2$ **A1**

b) $\mathbf{M} = \begin{pmatrix} -3\sqrt{3} & -3 \\ 3 & -3\sqrt{3} \end{pmatrix}$

So $\det \mathbf{M} = \begin{vmatrix} -3\sqrt{3} & -3 \\ 3 & -3\sqrt{3} \end{vmatrix} = 27 - (-9) = 36$ **M1**

> **Hint:** Rotation does not affect the scale factor, so det **M** = (enlargement scale factor)^2

So the area scale factor is 36 $\therefore$ positive scale factor of enlargement $= \sqrt{36} = 6$, so $k = 6$ **A1**

Using $k = 6$, $\mathbf{M} = \mathbf{ER} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 6\cos\theta & -6\sin\theta \\ 6\sin\theta & 6\cos\theta \end{pmatrix}$ **M1**

> **Hint:** Remember to get the matrices in the correct order (the transformations are always performed from right to left)

So $\begin{pmatrix} -3\sqrt{3} & -3 \\ 3 & -3\sqrt{3} \end{pmatrix} = \begin{pmatrix} 6\cos\theta & -6\sin\theta \\ 6\sin\theta & 6\cos\theta \end{pmatrix}$

So, for example, $6\cos\theta = -3\sqrt{3}$

> **Technique:** Equate the two forms of one of the elements of **M**, then check which angle is correct using another element of **M**

$\cos\theta = -\frac{\sqrt{3}}{2} \therefore \theta = 150°$ or $210°$ **M1**

Check using $6\sin\theta = 3$:

$6\sin 150° = 3$, $6\sin 210° = -3$ so $\theta = 150°$ **A1**

c) **M** maps $P(x, y)$ to $P'(a, b)$ so $\begin{pmatrix} -3\sqrt{3} & -3 \\ 3 & -3\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

The determinant of **M** is 36 and the inverse of **M** is $\mathbf{M}^{-1} = \frac{1}{36}\begin{pmatrix} -3\sqrt{3} & 3 \\ -3 & -3\sqrt{3} \end{pmatrix}$ **A1**

So $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{36}\begin{pmatrix} -3\sqrt{3} & 3 \\ -3 & -3\sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$ **M1**

$= \frac{1}{36}\begin{pmatrix} -3\sqrt{3}a + 3b \\ -3a - 3\sqrt{3}b \end{pmatrix} = \begin{pmatrix} -\sqrt{3}a/12 + b/12 \\ -a/12 - \sqrt{3}b/12 \end{pmatrix}$ **M1**

So the coordinates of P are $\left(-\frac{\sqrt{3}a}{12} + \frac{b}{12}, -\frac{a}{12} - \frac{\sqrt{3}b}{12}\right)$ **A1** **[12 Marks]**

7. $\mathbf{A} = \begin{pmatrix} -2 & a & 1 \\ 1 & 4 & c \\ 3 & b & 2 \end{pmatrix}$

$$\mathbf{A}^2 = \begin{pmatrix} -2 & a & 1 \\ 1 & 4 & c \\ 3 & b & 2 \end{pmatrix}\begin{pmatrix} -2 & a & 1 \\ 1 & 4 & c \\ 3 & b & 2 \end{pmatrix} = \begin{pmatrix} 10 & 4 & 12 \\ 14 & 11 & 25 \\ -2 & -3 & -1 \end{pmatrix}$$

Alternatively: You can use any other combination of rows and columns to isolate a, b and c

Using the element in the first row and the first column: $(-2)\times(-2)+a\times 1+1\times 3=10$

$4+a+3=10$

$7+a=10 \therefore a=3$ **A1**

Using the element in the third row and the first column: $3\times(-2)+b\times 1+2\times 3=-2$

$-6+b+6=-2$

$\therefore b=-2$ **A1**

Using the element in the second row and the first column: $1\times(-2)+4\times 1+c\times 3=14$

$-2+4+3c=14$

$2+3c=14 \therefore c=4$ **A1** **[3 Marks]**

Technique: Consider what happens to the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) under this transformation

8. a) Under a reflection in the plane $y = 0$, the points (1, 0, 0) and (0, 0, 1) do not move since they lie in this plane.
(0, 1, 0) reflects to (0, –1, 0) **M1**

Hence, a reflection in the plane $y = 0$ is given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **A1**

Alternatively: Learn that a rotation by angle θ anticlockwise about the x-axis is given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

b) Under a rotation of 90° about the x-axis, the point (1, 0, 0) does not move since it lies on this axis
(0, 1, 0) is rotated to (0, 0, 1) and (0, 0, 1) is rotated to (0, –1, 0) **M1**

Hence, a rotation of 90° about the x-axis is given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ **A1** **[4 Marks]**

Tip: If not specified by the question, you should assume that the rotation is performed anticlockwise about the axis facing towards the origin

9. $6x+3y+2z=10$
$3x+y+2z=4$
$-2y+az=-4$

a) Convert the coefficients to a matrix, $\begin{pmatrix} 6 & 3 & 2 \\ 3 & 1 & 2 \\ 0 & -2 & a \end{pmatrix}$

Find the determinant of this matrix in terms of a:

$$\begin{vmatrix} 6 & 3 & 2 \\ 3 & 1 & 2 \\ 0 & -2 & a \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & a \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 0 & a \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix} \quad \textbf{M1}$$

$= 6(a--4)-3(3a-0)+2(-6-0)$ **M1**

$= 6a+24-9a+-12$

$= 12-3a$ **M1**

Hint: The planes do not meet at a single point, so the matrix must be singular. Therefore the determinant is equal to zero.

There is no unique solution, so the matrix is singular, i.e. the determinant is equal to zero:

$12-3a=0 \therefore a=\frac{12}{3}=4$ **A1**

b) We have $6x+3y+2z=10$ **(1)**
$3x+y+2z=4$ **(2)**
$-2y+4z=-4$ **(3)**

Technique: Combine the simultaneous equations to eliminate one or more variables and determine whether the system is consistent

From **(1)** – 2 × **(2)**:
$y-2z=2$ **(4)** **M1**

(3) = –2 × **(4)**, i.e. **(3)** is just a linear multiple of **(4)** **A1**

So **(3)** and **(4)** are consistent with an infinite number of solutions **A1**

So the planes form a sheaf and meet along a line contained in all 3 planes **A1** **[8 Marks]**

TOTAL 58 MARKS

1. **Prove by induction that $\sum_{r=1}^{n}(2r-1)=n^2$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1}(2r-1)=2\times1-1=1$ and $n^2 = 1^2 = 1$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k}(2r-1)=k^2$ **M1**

When $n = k + 1$:

$$\sum_{r=1}^{k+1}(2r-1)=\sum_{r=1}^{k}(2r-1)+(2(k+1)-1)$$

$$=k^2+2(k+1)-1 \quad \textbf{M1}$$

$$=k^2+2k+1$$

$$=(k+1)^2 \quad \textbf{A1}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[5 Marks]**

2. **Prove by induction that $\sum_{r=1}^{n}(r^2+3r)=\frac{1}{3}n(n+1)(n+5)$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1}(r^2+3r)=1^2+3\times1=4$ and $\frac{1}{3}n(n+1)(n+5)=\frac{1}{3}\times1\times2\times6=4$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k}(r^2+3r)=\frac{1}{3}k(k+1)(k+5)$ **M1**

When $n = k + 1$:

$$\sum_{r=1}^{k+1}(r^2+3r)=\sum_{r=1}^{k}(r^2+3r)+((k+1)^2+3(k+1))$$

$$=\frac{1}{3}k(k+1)(k+5)+(k+1)^2+3(k+1) \quad \textbf{M1}$$

$$=\frac{1}{3}(k+1)(k(k+5)+3(k+1)+9)$$

$$=\frac{1}{3}(k+1)(k^2+8k+12) \quad \textbf{A1}$$

$$=\frac{1}{3}(k+1)(k+2)(k+6)$$

$$=\frac{1}{3}(k+1)((k+1)+1)((k+1)+5) \quad \textbf{A1}$$

So if the statement is true when $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

3. **Prove by induction that $f(n) = 7^n + 5$ is divisible by 6 for all positive integers n**

When $n = 1$: $f(1)=7^1+5=12=6\times2$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $f(k) = 7^k + 5$ is divisible by 6 **M1**

$f(k + 1) = 7^{k+1} + 5$

So $f(k+1)-f(k)=(7^{k+1}+5)-(7^k+5)$ **M1** ←

> **Alternatively:** For the inductive step you can show that $f(k + 1) = 7f(k) - 30$. Since $f(k)$ and 30 are both divisible by 6, $f(k + 1)$ will be too.

$$=7\times7^k-7^k$$

$$=7^k(7-1)=6\times7^k \quad \textbf{A1}$$

Hence $f(k+1)=f(k)+6\times7^k$ **A1**

Since $f(k)$ is divisible by 6 and 6×7^k is divisible by 6, their sum is divisible by 6

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

4. **Prove by induction that $\mathbf{A}^n = \begin{pmatrix} 1+4n & 4n \\ -4n & 1-4n \end{pmatrix}$ for every positive integer n**

$\mathbf{A} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$

When $n = 1$: left-hand side = $\mathbf{A}^1 = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$ and right-hand side = $\begin{pmatrix} 1+4\times1 & 4\times1 \\ -4\times1 & 1-4\times1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}^k = \begin{pmatrix} 1+4k & 4k \\ -4k & 1-4k \end{pmatrix}$ **M1**

$$\begin{aligned}\mathbf{A}^{k+1} &= \mathbf{A}^k\mathbf{A} \\ &= \begin{pmatrix} 1+4k & 4k \\ -4k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix} \quad \textbf{M1} \\ &= \begin{pmatrix} 5(1+4k)-4\times4k & 4(1+4k)-3\times4k \\ -4k\times5-4(1-4k) & -4k\times4-3(1-4k) \end{pmatrix} \quad \textbf{A1} \\ &= \begin{pmatrix} 5+4k & 4k+4 \\ -4k-4 & -3-4k \end{pmatrix} \\ &= \begin{pmatrix} 1+4(k+1) & 4(k+1) \\ -4(k+1) & 1-4(k+1) \end{pmatrix} \quad \textbf{A1}\end{aligned}$$

> **Alternatively:** You can write $\mathbf{A}^{k+1}$ as $\mathbf{A}^k \times \mathbf{A}$ or as $\mathbf{A} \times \mathbf{A}^k$. You will get the same result either way.

So if the statement is true for $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for every positive integer n **A1** **[6 Marks]**

5. **Prove by induction that $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1} 2^{r-1} = 2^{1-1} = 1$ and $2^n - 1 = 2^1 - 1 = 1$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k} 2^{r-1} = 2^k - 1$ **M1**

When $n = k + 1$:

$$\begin{aligned}\sum_{r=1}^{k+1} 2^{r-1} &= \sum_{r=1}^{k} 2^{r-1} + 2^{k+1-1} \\ &= 2^k - 1 + 2^k \quad \textbf{M1} \\ &= 2\times2^k - 1 \\ &= 2^{k+1} - 1 \quad \textbf{A1}\end{aligned}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[5 Marks]**

6. a) The fact that f(k) and $2k + 1$ are both odd does not mean their sum is odd **B1**
   This error is in the inductive step **B1**

   b) Any even number n serves as a counterexample, e.g. when $n = 2$, $f(2) = 2^2 = 4$, which is not odd **B1** **[3 Marks]**

7. $f(n) = 4^n - 3n + 2$

   a) $f(1) = 4^1 - 3\times1 + 2 = 3$ **M1**

   $f(2) = 4^2 - 3\times2 + 2 = 12$

   $f(3) = 4^3 - 3\times3 + 2 = 57$

   $f(4) = 4^4 - 3\times4 + 2 = 246$ **A1**

   b) $d = 3$ **B1**

c) **Prove your conjecture from part b) using induction**

We want to prove that f(n) is divisible by 3, where $f(n) = 4^n - 3n + 2$

When $n = 1$: $f(1) = 3 = 3 \times 1$ by part a) **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $f(k) = 4^k - 3k + 2$ is divisible by 3 **M1**

$$f(k+1) = 4^{k+1} - 3(k+1) + 2$$

So $f(k+1) - f(k) = \left(4^{k+1} - 3(k+1) + 2\right) - \left(4^k - 3k + 2\right)$ **M1** ←

> **Alternatively:** For the inductive step, you can show that f(k + 1) = 4f(k) + 9k – 9. Since f(k), 9k and 9 are all divisible by 3, f(k + 1) will be too.

$$= 4^{k+1} - 3k - 3 + 2 - 4^k + 3k - 2$$
$$= 4^{k+1} - 4^k - 3$$
$$= 4^k(4-1) - 3$$
$$= 3 \times 4^k - 3 = 3\left(4^k - 1\right)$$ **A1**

Hence $f(k+1) = f(k) + 3\left(4^k - 1\right)$ **A1**

Since f(k) is divisible by 3 and $3\left(4^k - 1\right)$ is divisible by 3, their sum is divisible by 3

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1**

[9 Marks]

8. **Prove by induction that $\mathbf{B}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for every natural number n**

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

When $n = 1$: left-hand side = $\mathbf{B}^1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and right-hand side = $\begin{pmatrix} 2^1 & 2^1 - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\mathbf{B}^k = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}$ **M1**

$$\mathbf{B}^{k+1} = \mathbf{B}^k\mathbf{B}$$
$$= \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$ **M1**
$$= \begin{pmatrix} 2^k \times 2 + \left(2^k - 1\right) \times 0 & 2^k \times 1 + \left(2^k - 1\right) \times 1 \\ 0 \times 2 + 1 \times 0 & 0 \times 1 + 1 \times 1 \end{pmatrix}$$ **A1**
$$= \begin{pmatrix} 2^{k+1} & 2 \times 2^k - 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$$ **A1**

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for every natural number n **A1** **[6 Marks]**

TOTAL 46 MARKS

Where they occur, λ and μ are scalar parameters.

1. a) $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda(4\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ **A1**

Allow equivalent equations in the correct form, such as $\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} 3+4\lambda \\ 2+\lambda \\ -1-3\lambda \end{pmatrix}$

b) $\frac{x-3}{4} = \frac{y-2}{1} = \frac{z+1}{-3}$ **A1** **[2 Marks]**

2. $\mathbf{a} = 7\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

a) $\mathbf{a}.\mathbf{b} = 7\times(-1) + 3\times 3 + (-4)\times(-5)$ **M1**

$= 22$ **A1**

b) If θ is the angle between **a** and **b**, then:

$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$|\mathbf{a}| = \sqrt{7^2 + 3^2 + (-4)^2} = \sqrt{74}$

$|\mathbf{b}| = \sqrt{(-1)^2 + 3^2 + (-5)^2} = \sqrt{35}$ **A1**

and $\mathbf{a}.\mathbf{b} = 22$ by part a)

So: $22 = \sqrt{74}\sqrt{35}\cos\theta$ **M1**

$\therefore \theta = \arccos\left(\frac{22}{\sqrt{74}\sqrt{35}}\right) = 64.3871...° = 64°$ to the nearest degree **A1** **[5 Marks]**

3. $l_1 : \mathbf{r} = (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $l_2 : x + 1 = y - 1 = z + 1$

a) l_2 has vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ **B1**

[Also allow $\mathbf{r} = (-\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$]

Tip: If the line L has Cartesian equation $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$, then it has vector equation $\mathbf{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

b) Using column vector notation, l_1 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

A general point A on l_1 has position vector $\begin{pmatrix} 2+2\lambda \\ 1+\lambda \\ -2+2\lambda \end{pmatrix}$

A general point B on l_2 has position vector $\begin{pmatrix} -1+\mu \\ 1+\mu \\ -1+\mu \end{pmatrix}$

So $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1+\mu \\ 1+\mu \\ -1+\mu \end{pmatrix} - \begin{pmatrix} 2+2\lambda \\ 1+\lambda \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} -3+\mu-2\lambda \\ \mu-\lambda \\ 1+\mu-2\lambda \end{pmatrix}$ **M1**

This vector needs to be perpendicular to l_1, so:

$\begin{pmatrix} -3+\mu-2\lambda \\ \mu-\lambda \\ 1+\mu-2\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$ **M1**

$\therefore (-3+\mu-2\lambda)\times 2 + (\mu-\lambda)\times 1 + (1+\mu-2\lambda)\times 2 = 0$ **M1**

$\therefore -6 + 2\mu - 4\lambda + \mu - \lambda + 2 + 2\mu - 4\lambda = 0$

$\therefore -4 + 5\mu - 9\lambda = 0$ **(1)**

The vector $\overrightarrow{AB}$ also needs to be perpendicular to l_2, so:

$\begin{pmatrix} -3+\mu-2\lambda \\ \mu-\lambda \\ 1+\mu-2\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$

$\therefore (-3+\mu-2\lambda)\times 1+(\mu-\lambda)\times 1+(1+\mu-2\lambda)\times 1=0$

$\therefore -3+\mu-2\lambda+\mu-\lambda+1+\mu-2\lambda=0$

$\therefore -2+3\mu-5\lambda=0$ **(2)**

So we have the simultaneous equations:

$5\mu-9\lambda=4$ **(1)**

$3\mu-5\lambda=2$ **(2)**

$5\times$**(1)** $-9\times$**(2)** gives:

$25\mu-45\lambda-27\mu+45\lambda=20-18$ **M1**

$\therefore -2\mu=2$

So $\mu=-1$

Substituting this into **(1)**, say, gives:

$5\times(-1)-9\lambda=4$

$\therefore -9\lambda=9$

So $\lambda=-1$

$\therefore \overrightarrow{AB}=\begin{pmatrix}-3+\mu-2\lambda\\ \mu-\lambda\\ 1+\mu-2\lambda\end{pmatrix}=\begin{pmatrix}-3-1+2\\ -1+1\\ 1-1+2\end{pmatrix}=\begin{pmatrix}-2\\0\\2\end{pmatrix}$ **A1**

Hence $\left|\overrightarrow{AB}\right|=\sqrt{(-2)^2+0^2+2^2}=\sqrt{8}$

So the shortest distance between the two lines is $\sqrt{8}$ $\left(\text{or } 2\sqrt{2}\right)$ **A1 [7 Marks]**

4. a) The line has equation $\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$

So a general point on the line has position vector $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+2\lambda\\2+3\lambda\\-3+\lambda\end{pmatrix}$

The plane has equation $x+3y-z=-10$, so the line and plane meet where:

$(1+2\lambda)+3(2+3\lambda)-(-3+\lambda)=-10$ **M1**

$1+2\lambda+6+9\lambda+3-\lambda=-10$

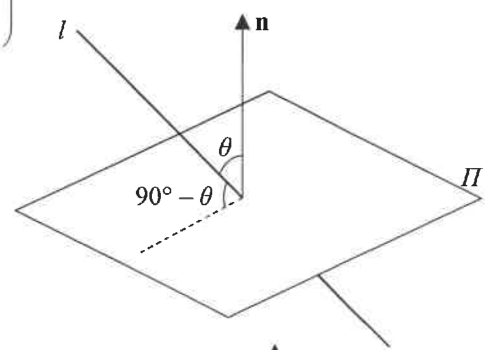
$10+10\lambda=-10$

$10\lambda=-20$

$\lambda=-2$ **A1**

Using the equation of the line, $\lambda=-2$ corresponds to the point $(1-4, 2-6, -3-2)=(-3,-4,-5)$ **A1**

b) A normal to the plane is $\begin{pmatrix}1\\3\\-1\end{pmatrix}$, and the line has direction $\begin{pmatrix}2\\3\\1\end{pmatrix}$



We have $\left|\begin{pmatrix}1\\3\\-1\end{pmatrix}\right|=\sqrt{1^2+3^2+(-1)^2}=\sqrt{11}$

$\left|\begin{pmatrix}2\\3\\1\end{pmatrix}\right|=\sqrt{2^2+3^2+1^2}=\sqrt{14}$

and $\begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix}=1\times 2+3\times 3+(-1)\times 1=10$

If the angle between these vectors is θ, then:

$\begin{pmatrix}1\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\1\end{pmatrix}=\left|\begin{pmatrix}1\\3\\-1\end{pmatrix}\right|\left|\begin{pmatrix}2\\3\\1\end{pmatrix}\right|\cos\theta$

Tip: If the plane Π has normal vector **n** and the line l has direction vector **v**, then $\mathbf{n.v}=|\mathbf{n}||\mathbf{v}|\cos\theta$, where θ is the angle between the line and the normal. The angle between the line and the plane itself is then $90°-\theta$.

So $10 = \sqrt{11}\sqrt{14}\cos\theta$ **M1**

and so $\theta = \arccos\left(\frac{10}{\sqrt{11}\sqrt{14}}\right) = 36.3101...°$ **A1**

Hence, the angle between the line and the plane is $90° - 36.3101...° = 53.6898...° = 53.7°$ (3 s.f.) (or 0.937 radians) **A1** **[6 Marks]**

5. $A(-2, 3, 7)$, $B(-1, 2, 7)$, $C(1, 6, 6)$

a) A is a point in the plane, and $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors that lie in the plane

A has position vector $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$$ **M1** for finding at least one vector in the plane

So an equation of the plane is $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$ **M1A1**

Allow any equation in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ where $\mathbf{a}$ is the position vector of a point in the plane and $\mathbf{b}$ and $\mathbf{c}$ are vectors that lie in the plane

b) **Show that A, B, C and D are not coplanar**

By part a), the points A, B and C lie in the plane with equation $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}$

If A, B, C and D are coplanar, then D also lies in this plane

D has position vector $\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$

So if D lies in the plane, then there are values of λ and μ such that $\begin{pmatrix} -2+\lambda+3\mu \\ 3-\lambda+3\mu \\ 7-\mu \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$ **M1**

This gives the three simultaneous equations:

$\lambda + 3\mu = 6$ **(1)**

$-\lambda + 3\mu = 1$ **(2)**

$-\mu = -7$ **(3)**

From **(3)** we have $\mu = 7$

Substituting this into **(1)** gives $\lambda + 3\times 7 = 6$, so $\lambda = 6 - 21 = -15$ **A1**

Substituting $\lambda = -15$, $\mu = 7$ into **(2)** gives $15 + 3\times 7 = 36 \neq 1$

So the three equations are inconsistent. There are no solutions to all three equations, and so D does not lie in the plane. Hence the four points are not coplanar. **A1** **[6 Marks]**

6. a) The line l_1 passes through the points with position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}$

A direction vector for l_1 is $\mathbf{b} - \mathbf{a} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$ **M1**

So l_1 has vector equation $\mathbf{r} = \begin{pmatrix} 5 \\ 6 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$ **A1**

Allow any equation in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ where $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$ is a non-zero multiple of the vector $-2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

b) **Show that the lines l_1 and l_2 are skew**

Using column vector notation, l_2 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$

A general point on l_1 has position vector $\begin{pmatrix} 5-2\lambda \\ 6-3\lambda \\ -2+3\lambda \end{pmatrix}$

A general point on l_2 has position vector $\begin{pmatrix} 3 \\ 2\mu \\ -2+3\mu \end{pmatrix}$

So if l_1 and l_2 intersect, then $\begin{pmatrix} 5-2\lambda \\ 6-3\lambda \\ -2+3\lambda \end{pmatrix} = \begin{pmatrix} 3 \\ 2\mu \\ -2+3\mu \end{pmatrix}$ **M1**

This gives the simultaneous equations:

$5-2\lambda = 3$ which rearrange to: $2\lambda = 2$ **(1)**

$6-3\lambda = 2\mu$ $\quad 3\lambda + 2\mu = 6$ **(2)**

$-2+3\lambda = -2+3\mu$ $\quad -3\lambda + 3\mu = 0$ **(3)**

(1) gives $\lambda = 1$

Substituting this into **(2)** gives $3 \times 1 + 2\mu = 6$, so $\mu = \frac{6-3}{2} = \frac{3}{2}$ **A1**

Substituting $\lambda = 1,\ \mu = \frac{3}{2}$ into **(3)** gives $-3\times1+3\times\frac{3}{2} = \frac{3}{2} \neq 0$, so the equations are inconsistent. Hence l_1 and l_2 do not intersect. **A1**

The direction of l_1 is $\begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$. The direction of l_2 is $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \neq k \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$ for any $k \in \mathbb{R}$, so the lines are not parallel. **A1**

Hence the two lines are skew **[6 Marks]**

7. $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{b} = -5\mathbf{j} + p\mathbf{k}$

a and **b** are perpendicular, so $\mathbf{a}.\mathbf{b} = 0$

$\mathbf{a}.\mathbf{b} = 3\times0 + (-2)\times(-5) + (-5)\times p = 10 - 5p$ **A1**

So $10 - 5p = 0$ **M1**

Hence $p = \frac{10}{5} = 2$ **A1** **[3 Marks]**

8. $l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ -3 \\ 19 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$, $l_2 : \mathbf{r} = \begin{pmatrix} 16 \\ -2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -8 \\ 6 \\ -2 \end{pmatrix}$

A general point on l_1 has position vector $\begin{pmatrix} -1+\lambda \\ -3+2\lambda \\ 19-4\lambda \end{pmatrix}$

A general point on l_2 has position vector $\begin{pmatrix} 16-8\mu \\ -2+6\mu \\ 2-2\mu \end{pmatrix}$

So the lines meet when $\begin{pmatrix} -1+\lambda \\ -3+2\lambda \\ 19-4\lambda \end{pmatrix} = \begin{pmatrix} 16-8\mu \\ -2+6\mu \\ 2-2\mu \end{pmatrix}$ **M1**

This gives the simultaneous equations:

$-1+\lambda = 16-8\mu$ which rearrange to: $\lambda + 8\mu = 17$ **(1)**

$-3+2\lambda = -2+6\mu$ $\quad 2\lambda - 6\mu = 1$ **(2)**

$19-4\lambda = 2-2\mu$ $\quad -4\lambda + 2\mu = -17$ **(3)**

Equation **(1)** gives $\lambda = 17 - 8\mu$. Substituting this into equation **(2)** gives:

$2(17-8\mu) - 6\mu = 1$ **M1**

$\therefore -22\mu = -33$

So $\mu = \frac{-33}{-22} = \frac{3}{2}$

Hence $\lambda = 17 - 8 \times \frac{3}{2} = 5$ **A1**

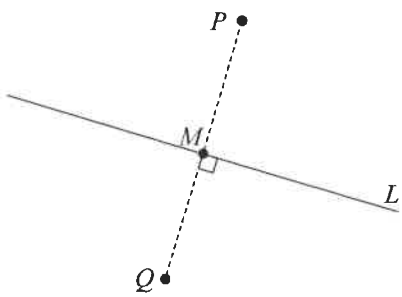
Substituting $\lambda = 5,\ \mu = \frac{3}{2}$ into equation **(3)** gives:

$-4 \times 5 + 2 \times \frac{3}{2} = -20 + 3 = -17$

So the equations are consistent, and the lines do intersect where $\lambda = 5$ and $\mu = \frac{3}{2}$ **A1**

Using the equation for l_1, this corresponds to the point (–1 + 5, –3 + 10, 19 – 20) = (4, 7, –1) **A1** **[5 Marks]**

9. $L : \mathbf{r} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix},\ P = (-1, -5, 4)$



Let Q be the reflection of P in the line L. Then PQ must be perpendicular to L and its midpoint M must lie on L (see diagram to the right).

P has position vector $\begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix}$

M lies on L so its position vector is $\begin{pmatrix} 2-\lambda \\ 5-3\lambda \\ -4+3\lambda \end{pmatrix}$ for some value of λ

So $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \begin{pmatrix} 2-\lambda \\ 5-3\lambda \\ -4+3\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix} = \begin{pmatrix} 3-\lambda \\ 10-3\lambda \\ -8+3\lambda \end{pmatrix}$ **M1**

This vector must be perpendicular to L, which has direction $\begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix}$, so:

$\begin{pmatrix} 3-\lambda \\ 10-3\lambda \\ -8+3\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -3 \\ 3 \end{pmatrix} = 0$ **M1**

$\therefore (3-\lambda)\times(-1) + (10-3\lambda)\times(-3) + (-8+3\lambda)\times 3 = 0$ **M1**

$$-3 + \lambda - 30 + 9\lambda - 24 + 9\lambda = 0$$

$$-57 + 19\lambda = 0$$

$$\lambda = \frac{57}{19}$$

$\lambda = 3$ **A1**

So $\overrightarrow{PM} = \begin{pmatrix} 3-\lambda \\ 10-3\lambda \\ -8+3\lambda \end{pmatrix} = \begin{pmatrix} 3-3 \\ 10-9 \\ -8+9 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ **M1**

Since M is the midpoint of PQ, we have $\overrightarrow{PQ} = 2\overrightarrow{PM} = 2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$

and the position vector of Q is given by $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 6 \end{pmatrix}$

So Q has coordinates (–1, –3, 6) **A1** **[6 Marks]**

10. Let the point T represent the air traffic control tower, which has position vector $\begin{pmatrix} 9 \\ -1 \\ 0 \end{pmatrix}$

The aeroplane flies in a straight line L from A with position vector $\begin{pmatrix} -15 \\ 7 \\ 0 \end{pmatrix}$ to B with position vector $\begin{pmatrix} 19 \\ -9 \\ 2 \end{pmatrix}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 19 \\ -9 \\ 2 \end{pmatrix} - \begin{pmatrix} -15 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 34 \\ -16 \\ 2 \end{pmatrix} \quad \textbf{A1}$$

So an equation for L is $\mathbf{r} = \begin{pmatrix} -15 \\ 7 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 34 \\ -16 \\ 2 \end{pmatrix}$ **M1**

A general point P on the line L has position vector $\begin{pmatrix} -15+34\lambda \\ 7-16\lambda \\ 2\lambda \end{pmatrix}$

So $\overrightarrow{PT} = \overrightarrow{OT} - \overrightarrow{OP} = \begin{pmatrix} 9 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -15+34\lambda \\ 7-16\lambda \\ 2\lambda \end{pmatrix} = \begin{pmatrix} 24-34\lambda \\ -8+16\lambda \\ -2\lambda \end{pmatrix}$

If P is the closest point on L to T, then $\overrightarrow{PT}$ is perpendicular to L, which has direction $\begin{pmatrix} 34 \\ -16 \\ 2 \end{pmatrix}$, so:

$$\begin{pmatrix} 24-34\lambda \\ -8+16\lambda \\ -2\lambda \end{pmatrix} \cdot \begin{pmatrix} 34 \\ -16 \\ 2 \end{pmatrix} = 0 \quad \textbf{M1}$$

$$\therefore (24-34\lambda)\times 34 + (-8+16\lambda)\times(-16) + (-2\lambda)\times 2 = 0 \quad \textbf{M1}$$

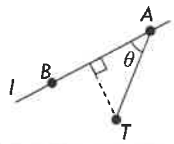
$$816 - 1156\lambda + 128 - 256\lambda - 4\lambda = 0$$

$$944 - 1416\lambda = 0$$

$$\lambda = \frac{944}{1416}$$

$$\lambda = \frac{2}{3} \quad \textbf{A1}$$

Alternatively: If A and B are two points on L, then the shortest distance from T to L is $\left|\overrightarrow{AT}\right| \sin\theta$ where θ is the acute angle between $\overrightarrow{AT}$ and L and is given by $\cos\theta = \left|\dfrac{\overrightarrow{AT}.\overrightarrow{AB}}{\left|\overrightarrow{AT}\right|\left|\overrightarrow{AB}\right|}\right|$, as shown below:



$$\text{So } \overrightarrow{PT} = \begin{pmatrix} 24-34\lambda \\ -8+16\lambda \\ -2\lambda \end{pmatrix} = \begin{pmatrix} 24-34\times\frac{2}{3} \\ -8+16\times\frac{2}{3} \\ -2\times\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ \frac{8}{3} \\ -\frac{4}{3} \end{pmatrix} \quad \textbf{A1}$$

$$\text{Hence } \left|\overrightarrow{PT}\right| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{8}{3}\right)^2 + \left(-\frac{4}{3}\right)^2} = \sqrt{\frac{32}{3}} = 3.26598...$$

So the shortest distance from the aeroplane to the air traffic control tower is 3.26598... km **A1**
3.26598... > 3 so the airport is not responsible for the aeroplane during this flight **A1** **[8 Marks]**

TOTAL 54 MARKS