Complex Numbers II – Test A (40 mins)

✓**Fundamentals** Challenge Expert

Subtopics: Exponential form of complex numbers, multiplying and dividing complex numbers, de Moivre's theorem, trigonometric identities, sums of series, n^{th} roots of a complex number, solving geometric problems

1. Express each of the following in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. Give **exact** values for r and θ where possible, or to **3 significant figures** otherwise:
a) -4 b) $5i$ c) $1-2i$ **[4]**

2. Express each of the following in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$:
a) $e^{\frac{\pi}{5}i}$ b) $6e^{\frac{12\pi}{5}i}$ c) $e^{\frac{-15\pi}{4}i}$ **[5]**

3. Use **Euler's relation**, $e^{i\theta} = \cos\theta + i\sin\theta$, to show that $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$ **[3]**

4. Express $4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 7\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$, showing your working. **[3]**

5. Express $\dfrac{6e^{\frac{5\pi}{2}i}}{3e^{\frac{3\pi}{2}i}}$ in the form $a + bi$, where $a,\ b \in \mathbb{R}$, showing your working. **[4]**

6. Use **de Moivre's theorem** to express $\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^{10}$ in the form $a + bi$, where $a,\ b \in \mathbb{R}$ **[3]**

7. Use **de Moivre's theorem** to prove that $\cos 2\theta \equiv 2\cos^2\theta - 1$ **[5]**

8. Given that $z = \cos\theta + i\sin\theta$, use the results $z^n - \frac{1}{z^n} = 2i\sin n\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$ to prove that $\sin^2\theta \equiv \frac{1}{2}(1 - \cos 2\theta)$ **[5]**

9. a) Use de Moivre's theorem to find $\left(1 + i\sqrt{3}\right)^6$, showing your working **[3]**
b) Hence show that $\sum_{k=0}^{5}\left(1 + i\sqrt{3}\right)^k = -21i\sqrt{3}$ **[3]**

Hint: The sum of the first n terms of a geometric series with first term a and common ratio r is $S_n = a\left(\frac{r^n - 1}{r - 1}\right)$

10. a) Solve the equation $z^4 = 1$ **using de Moivre's theorem** **[6]**
b) Illustrate the roots of this equation on an Argand diagram **[1]**
c) Find the **centre** and **radius** of the circle on which these four roots lie **[2]**
d) Show that these roots can be written as $1,\ \omega,\ \omega^2$ and ω^3, where $1 + \omega + \omega^2 + \omega^3 = 0$, for a suitable complex number ω **[4]**

11. Given that a **regular pentagon** whose centre is the origin has one vertex at (2, 0), find the coordinates of the other vertices to **3 significant figures**.
(The centre of a regular polygon is considered to be the centre of the circle that passes through all of the polygon's vertices.) **[6]**

TOTAL 57 MARKS

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Series – Test A (30 mins)

✓**Fundamentals** Challenge Expert

Subtopics: The method of differences, higher derivatives, Maclaurin series, series expansions of compound functions

1. a) Show that $\frac{1}{r}-\frac{1}{r+1}\equiv\frac{1}{r(r+1)}$ **[1]**

   b) Hence, by using part a) and writing out all the terms, show that $\sum_{r=1}^{5}\frac{1}{r(r+1)}=\frac{5}{6}$ **[3]**

2. a) Express $\frac{2}{(r+1)(r+2)}$ in the form $\frac{A}{r+1}+\frac{B}{r+2}$, where A and B are constants to be found. **[2]**

   b) Hence show that $\sum_{r=1}^{12}\frac{2}{(r+1)(r+2)}=\frac{6}{7}$, using the **method of differences**. **[3]**

   c) Find an expression for $\sum_{r=1}^{n}\frac{2}{(r+1)(r+2)}$, simplifying your answer as far as possible. **[2]**

3. State one reason why it may not be possible to write a function $f(x)$ as a Maclaurin series expansion. **[1]**

4. a) For $f(x)=\sqrt[3]{1+x}$, find $f'(x)$, $f''(x)$, and $f'''(x)$ **[5]**

   b) Hence show that the Maclaurin series expansion for $f(x)$ is $f(x)=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\ldots$ **[2]**

5. $f(x)=\mathrm{e}^{4x}$ and $g(x)=\ln(1+3x)$

   a) Use standard Maclaurin series results to express $f(x)$ and $g(x)$ as a series of ascending powers of x, **up to and including** the terms in x^4 **[4]**

   b) State the range of values of x for which each series expansion is valid **[2]**

6. Use the Maclaurin series of $\ln(1+x)$ to show that $\ln\left(\frac{1+x}{1-2x}\right)=3x+\frac{3}{2}x^2+ax^3+bx^4+\ldots$,

   where a and b are constants to be found. State the range of values of x for which the expansion of $\ln\left(\frac{1+x}{1-2x}\right)$ is valid. **[5]**

   Hint: Use the logarithm law $log\frac{a}{b}=log\,a-log\,b$

7. a) Find the first **three non-zero** terms in the Maclaurin series expansions of cos $3x$ and sin $2x$ **[4]**

   b) Hence show that $2x\cos 3x-5\sin 2x=-8x-\frac{7}{3}x^3+\frac{65}{12}x^5+\ldots$ **[2]**

TOTAL 36 MARKS

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Further Calculus – Test A (45 mins) ✓ **Fundamentals** Challenge Expert

Subtopics: Improper integrals, the mean value of a function, differentiating inverse trigonometric functions, integrating with inverse trigonometric functions, integrating using partial fractions, volumes of revolution around the x-axis and the y-axis, volumes of revolution of parametrically defined curves, modelling with volumes of revolution

1. a) Find $\int \frac{1}{x^2}\,\mathrm{d}x$ **[1]**

   b) Hence show that $\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ **converges**, and find its value **[3]**

2. a) Find $\int \frac{1}{16+x^2}\,\mathrm{d}x$ **[1]**

   b) Find $\int \frac{1}{16+9x^2}\,\mathrm{d}x$, giving your answer in the form $A\arctan(Bx)+c$, where A and B are constants to be found, and c is an arbitrary constant **[3]**

3. a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{9-x^2}$ **[1]**

   b) Hence evaluate $\int_0^3 \frac{2x}{\sqrt{9-x^2}}\,\mathrm{d}x$ **[5]**

4. Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ **[4]**

5. Find $\int_1^4 3x\,\mathrm{d}x$, and hence find the **exact mean value** of $f(x)=3x$ over the interval $[1,\ 4]$ **[5]**

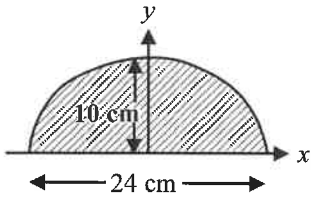
6. Use the substitution $x = \frac{1}{2}\cos\theta$ to find $\int \frac{-1}{\sqrt{1-4x^2}}\,\mathrm{d}x$ in the form $A\arccos(Bx)+c$, where A and B are constants to be found and c is an arbitrary constant. **[4]**

7. a) Show that $\frac{2x^2-6x+17}{(2x-3)(x^2+4)} \equiv \frac{A}{2x-3}+\frac{B}{x^2+4}$, where A and B are constants to be found. **[3]**

   b) Hence show that $\int \frac{2x^2-6x+17}{(2x-3)(x^2+4)}\,\mathrm{d}x = C\ln|2x-3| + D\arctan\left(\frac{x}{2}\right)+c$, where C and D are constants to be found, and c is an arbitrary constant. **[3]**

8. Find the **exact volume** of the solid formed when the curve $y=\sqrt{\sin 2x}$ is rotated 2π radians about the x-axis between $x=0$ and $x=\frac{\pi}{4}$ **[4]**

9. The diagram shows the cross section of a cake. The cake can be modelled as the solid formed by a revolution of 2π radians about the y-axis of the region bounded by the axes and the curve with parametric equations $x=12\cos\theta,\ y=10\sin\theta$ for $0\le\theta\le\frac{\pi}{2}$. Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, find the **exact volume** of the cake predicted by the model. **[7]**



TOTAL 44 MARKS

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Polar Coordinates – Test A (55 mins)

✓ **Fundamentals** Challenge Expert

Subtopics: Polar coordinates and equations, sketching curves, area enclosed by a polar curve, tangents to polar curves

1. Find **polar coordinates** (r, θ), where $-\pi < \theta \leq \pi$, of the points with the following Cartesian coordinates. Give θ to **3 significant figures** or **in terms of π** where appropriate:
   a) $(4, 3)$ b) $(-12, 5)$ c) $(-7, 0)$ d) $\left(1, -\sqrt{3}\right)$ **[8]**

2. Find the **Cartesian coordinates** (x, y) of the points with the following polar coordinates. Give x and y in **surd form** where appropriate.
   a) $(5, \pi)$ b) $\left(5, \frac{3\pi}{2}\right)$
   c) $\left(2, \frac{7\pi}{4}\right)$ d) $\left(4, \frac{2\pi}{3}\right)$ **[8]**

3. Find **Cartesian equations** for the following curves:
   a) $r = 5$ b) $r\cos\theta = 2$
   c) $r = 3\operatorname{cosec}\theta$ d) $r^2 = \cos\theta\sin\theta$ **[8]**

4. On separate diagrams, **sketch** the following curves for $0 \leq \theta < 2\pi$:
   a) $r = 1$ b) $\theta = \frac{\pi}{4}$ c) $r = \theta$ **[6]**

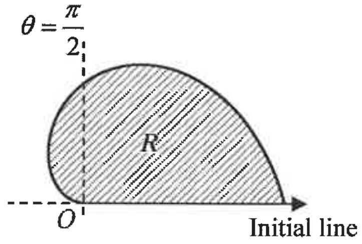
5. a) Copy and complete the following table relating values of θ to values of $r = 1 + \cos\theta$, where the values of r are given to **2 decimal places** where appropriate: **[1]**

$\boldsymbol{\theta}$	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\boldsymbol{r = 1 + \cos\theta}$	2	1.92		1.38		0.62	0.29		

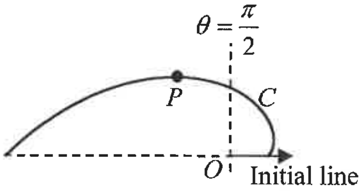
   b) By considering the symmetry of the curve, **sketch** the curve $r = 1 + \cos\theta$ for $-\pi \leq \theta < \pi$ **[2]**

6. **Sketch** the curve $r^2 = \sin\theta$ for $0 \leq \theta \leq \pi$ **[2]**

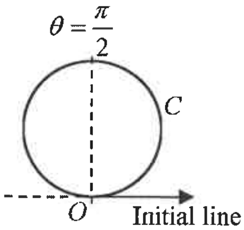
7. The finite region R is bounded by the curve with equation $r = \pi - \theta$ and the half-lines $\theta = 0$ and $\theta = \pi$, as shown to the right. Use **calculus** to find the **exact** area of R. **[3]**



8. The curve C sketched to the right has equation $r = e^{\theta}$ for $0 \leq \theta \leq \pi$. Find the polar coordinates of the point P on C where the tangent is **parallel** to the initial line. Give your coordinates to **3 significant figures**. **[4]**



9. The curve C sketched to the right has equation $r = 2\sin\theta$. Find the **exact** polar coordinates of the points on C where the tangent is **perpendicular** to the initial line. **[5]**



TOTAL 47 MARKS

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Hyperbolic Functions – Test A (40 mins)

✓ **Fundamentals** Challenge Expert

Subtopics: Hyperbolic functions, inverse hyperbolic functions, identities and equations, differentiating hyperbolic functions, integrating hyperbolic functions

1. **Use the exponential definitions** of the hyperbolic functions to find, correct to **2 decimal places**, the values of the following functions. Show your working.

   a) $\cosh 2$ b) $\sinh \frac{1}{3}$ c) $\tanh(-1)$ **[6]**

2. **Use the exponential definition** of $\sinh x$ to find the **exact** value of $\sinh(\ln 3)$, showing your working. **[3]**

3. a) On the **same diagram**, sketch the graphs of $y = \cosh x$ and $y = \cosh 5x$ **[2]**

   b) State the **range** of $f(x) = \cosh 5x$, where $x \in \mathbb{R}$ **[1]**

4. On the **same diagram**, sketch the graphs of $y = \tanh x$ and $y = \text{artanh}\, x$, indicating any **asymptotes**. **[4]**

5. Express each of the following as a **natural logarithm**, simplifying each answer as far as possible:

   a) $\text{arcosh}\, 4$ b) $\text{arsinh}\sqrt{3}$ c) $\text{artanh}\, \frac{1}{4}$ **[6]**

6. Find the following integrals, giving your answers in terms of inverse hyperbolic functions:

   a) $\int \frac{7}{\sqrt{x^2+1}}\, dx$ b) $\int \frac{3}{\sqrt{x^2-1}}\, dx$ **[2]**

7. Differentiate each of the following with respect to x, leaving hyperbolic functions in your answers:

   a) $\cosh 4x$ b) $\frac{1}{3}\tanh 3x$ c) $x \sinh 2x$ d) $\ln 2x \cosh 2x$ **[6]**

8. Using the results $\cosh 2A \equiv \cosh^2 A + \sinh^2 A$ and $\sinh 2A \equiv 2\sinh A \cosh A$, **show that**

   $\tanh 2A \equiv \frac{2\tanh A}{1+\tanh^2 A}$ **[3]**

9. Solve the equation $\cosh x + 2\sinh x = 1$ for real values of x **[6]**

10. Given that $y = 2\cosh 4x + 3\sinh 4x$, prove that $\frac{d^2y}{dx^2} = ny$, where n is a constant to be found. **[4]**

11. Find $\int \cosh^3 2x \sinh 2x\, dx$, giving your answer in **hyperbolic form.** **[2]**

12. Using the exponential definition of $\cosh x$, find $\int e^{3x} \cosh x\, dx$, giving your answer in **exponential form.** **[3]**

13. By writing $\cosh^2 x$ in terms of $\cosh 2x$, find $\int \cosh^2 x\, dx$ **[3]**

14. Use the substitution $x = 5\sinh u$ to find $\int \frac{1}{\sqrt{x^2+25}}\, dx$ **[4]**

TOTAL 55 MARKS

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Differential Equations – Test A (45 mins)

✓**Fundamentals** Challenge Expert

Subtopics: First- and second-order differential equations, boundary conditions, modelling, simple harmonic motion, damped and forced harmonic motion, coupled first-order simultaneous differential equations

1. Find the **integrating factor** for each of the following differential equations:

   a) $\frac{\mathrm{d}y}{\mathrm{d}x}+2xy=1$ b) $\frac{\mathrm{d}y}{\mathrm{d}x}+y=x$ **[4]**

2. Use the integrating factor x to find the **general solution** of $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=3x$.

   Give your answer in the form $y = \mathrm{f}(x)$. **[4]**

3. Find the **general solutions** to the following differential equations, where y is a function of x:
   a) $y''-3y'+2y=0$ **[3]**
   b) $y''-2y'+y=0$ **[3]**
   c) $y''-2y'+2y=0$ **[3]**

4. a) Find the **complementary function** for the differential equation $y''-4y'-5y=3\mathrm{e}^{-x}$ **[3]**
   b) Explain why $\lambda\mathrm{e}^{-x}$ (where $\lambda\in\mathbb{R}$) is **not** a suitable form for the particular integral for this equation **[1]**

5. A particle is moving in a straight line. Its displacement from the point O at time t seconds is x m. At time t the velocity of the particle is given by $(2t+\cos t)$ m s^{-1}. When $t=\pi$ s, the particle is at the point O.
   How far from O is the particle when $t=2\pi$ s? Give your answer **exactly**. **[4]**

6. A particle is moving with simple harmonic motion around the centre of oscillation, O. After t seconds, the particle's displacement from O, x m, is given by the differential equation $\ddot{x}=-\frac{1}{4}x$.

   When $t=0$, the particle is at O and is moving with velocity $\frac{1}{2}$ m s^{-1}.

   a) Show that the particle's velocity, v, satisfies $v\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1}{4}x$ **[2]**
   b) Hence show that v satisfies the equation $kv^2=1-x^2$, where k is an **integer** to be found **[3]**

7. The displacement from O at time t of a particle moving with forced harmonic motion is denoted by x and satisfies the differential equation $\ddot{x}+4\dot{x}+9x=20\cos t$.
   Verify that $x=\sin t+2\cos t$ is a **particular solution** to this equation. **[4]**

8. Consider the coupled differential equations:

   $$\frac{\mathrm{d}x}{\mathrm{d}t}=2y$$

   $$\frac{\mathrm{d}y}{\mathrm{d}t}=3x+y$$

   a) Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{\mathrm{d}x}{\mathrm{d}t}-6x=0$ **[4]**
   b) Hence find a **general solution** for x in terms of t **[3]**

TOTAL 41 MARKS

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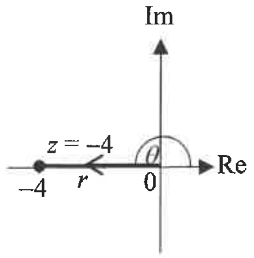
Solutions to Complex Numbers II – Test A

✓**Fundamentals** Challenge Expert

1. a)



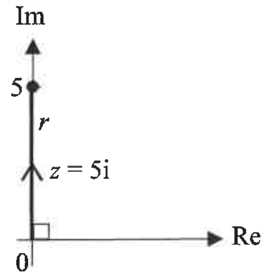
Technique: You can write a complex number z in the form $z = re^{i\theta}$ using the facts that $r = |z|$ and $\theta = \arg z$. Sketching an Argand diagram will help you to find r and θ.

$r = |z| = \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4, \ \theta = \arg z = \pi$

$\therefore z = 4e^{\pi i}$ **A1**

Tip: Learn how to use the Complex menu on your calculator to convert between the different forms of complex numbers

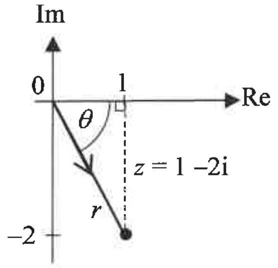
b)



$r = |z| = \sqrt{0^2 + 5^2} = \sqrt{25} = 5, \ \theta = \arg z = \frac{\pi}{2}$

$\therefore z = 5e^{\frac{\pi}{2}i}$ **A1**

c)



$r = |z| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

$\theta = \arg z = -\arctan\left(\frac{2}{1}\right) = -1.10714... = -1.11$ (3 s.f.) **M1**

$\therefore z = \sqrt{5}e^{-1.11i}$ **A1** **[4 Marks]**

2. a) $e^{\frac{\pi}{5}i} = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$ $\left(\text{also allow } 1\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)\right)$ **A1**

b) $\frac{12\pi}{5} - 2\pi = \frac{2\pi}{5}$ **M1**

So $6e^{\frac{12\pi}{5}i} = 6e^{\frac{2\pi}{5}i} = 6\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$ **A1**

Technique: Use the facts that $\cos\theta = \cos(\theta \pm 2\pi)$ and $\sin\theta = \sin(\theta \pm 2\pi)$ to find a value of θ in the interval $-\pi < \theta \leq \pi$

c) $-\frac{15\pi}{4} + 2\pi = -\frac{7\pi}{4}$

$-\frac{7\pi}{4} + 2\pi = \frac{\pi}{4}$ **M1**

Hint: You may have to add or subtract a multiple of 2π in order to find a value of θ in the given interval

So $e^{-\frac{15\pi}{4}i} = e^{\frac{\pi}{4}i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ $\left(\text{also allow } 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)$ **A1** **[5 Marks]**

3. **Show that** $\boldsymbol{\cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore e^{-i\theta} = e^{i(-\theta)} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$ **M1**

So $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$ **M1**

$\therefore \cos\theta = \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)$ **A1** **[3 Marks]**

4. $4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\times 7\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$ is of the form z_1z_2, with $z_1 = 4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $z_2 = 7\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

So $z_1z_2 = 4e^{\frac{\pi}{3}i}\times 7e^{\frac{\pi}{2}i}$ **M1**

$= (4\times 7)e^{i\left(\frac{\pi}{3}+\frac{\pi}{2}\right)}$ **M1**

$= 28e^{\frac{5\pi}{6}i}$ **A1** **[3 Marks]**

Technique: First write both numbers in the form $re^{i\theta}$. Then multiply the moduli and add the arguments to find the result of the multiplication.

5. $\dfrac{6e^{\frac{5\pi}{2}i}}{3e^{\frac{3\pi}{2}i}}$ is of the form $\dfrac{z_1}{z_2}$, with $z_1 = 6e^{\frac{5\pi}{2}i}$ and $z_2 = 3e^{\frac{3\pi}{2}i}$

So $\dfrac{z_1}{z_2} = \dfrac{6}{3}e^{i\left(\frac{5\pi}{2}-\frac{3\pi}{2}\right)}$ **M1**

$= 2e^{\pi i}$ **M1**

$= 2(\cos\pi + i\sin\pi)$ **M1**

$= 2(-1+0i) = -2+0i = -2$ **A1** **[4 Marks]**

Technique: First find the result of the division in the form $re^{i\theta}$ by dividing the moduli and subtracting the arguments. Then write the result in the form $a + bi$.

6. $\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)^{10} = 1^{10}\left(\cos\left(10\times\frac{5\pi}{6}\right)+i\sin\left(10\times\frac{5\pi}{6}\right)\right)$ **M1**

$= \cos\frac{25\pi}{3} + i\sin\frac{25\pi}{3}$ **M1**

$= \frac{1}{2}+\frac{\sqrt{3}}{2}i$ **A1** **[3 Marks]**

Hint: de Moivre's theorem states that $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

7. **Prove that** $\boldsymbol{\cos 2\theta \equiv 2\cos^2\theta - 1}$

$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$ by de Moivre's theorem **M1**

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2\cos\theta\times i\sin\theta + (i\sin\theta)^2$ **M1**

$= \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta$

$= \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta$ **A1**

Equating real parts, $\cos 2\theta \equiv \cos^2\theta - \sin^2\theta$

$\equiv \cos^2\theta - \left(1-\cos^2\theta\right)$ **M1**

$\equiv 2\cos^2\theta - 1$

$\therefore \cos 2\theta \equiv 2\cos^2\theta - 1$ **A1** **[5 Marks]**

Technique: Use the identity $\cos^2\theta + \sin^2\theta \equiv 1$

8. **Given that $z = \cos\theta + \mathrm{i}\sin\theta$, use the results $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$ to prove that**

$\sin^2\theta \equiv \frac{1}{2}(1-\cos 2\theta)$

For $n = 1$, $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$

So $\left(z-\frac{1}{z}\right)^2 = (2\mathrm{i}\sin\theta)^2 = 4\mathrm{i}^2\sin^2\theta = -4\sin^2\theta$ **M1**

Also $\left(z-\frac{1}{z}\right)^2 = z^2 + 2z\left(-\frac{1}{z}\right) + \left(-\frac{1}{z}\right)^2$ **M1**

$= z^2 - 2 + \frac{1}{z^2}$

$= \left(z^2 + \frac{1}{z^2}\right) - 2$ **M1**

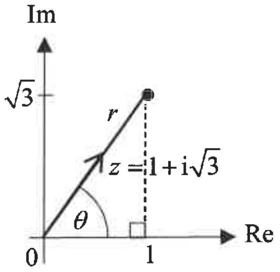
Technique: Group terms to the same power of z and rewrite the expression in terms of cos $n\theta$

$= 2\cos 2\theta - 2$ by using $n = 2$ **M1**

$\therefore -4\sin^2\theta \equiv 2\cos 2\theta - 2$

$\therefore \sin^2\theta \equiv \frac{2}{-4}\cos 2\theta - \frac{2}{-4} \equiv \frac{1}{2}(1-\cos 2\theta)$ **A1** **[5 Marks]**

9. a) Let $1 + \mathrm{i}\sqrt{3} = z$



$r = |z| = \sqrt{1 + (\sqrt{3})^2} = \sqrt{4} = 2$

$\theta = \arg z = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$ **M1**

$\therefore z = 2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$ **A1**

Technique: Rewrite z in the form $r\mathrm{e}^{\mathrm{i}\theta}$ to make it easier to find z^6. Sketching an Argand diagram will make it easier to do this.

So $z^6 = 2^6\mathrm{e}^{\left(6\times\frac{\pi}{3}\right)\mathrm{i}} = 64\mathrm{e}^{2\pi\mathrm{i}} = 64(\cos 2\pi + \mathrm{i}\sin 2\pi) = 64$ **A1**

b) **Show that $\sum_{k=0}^{5}(1+\mathrm{i}\sqrt{3})^k = -21\mathrm{i}\sqrt{3}$**

$\sum_{k=0}^{5}(1+\mathrm{i}\sqrt{3})^k = \sum_{k=0}^{5} z^k = z^0 + z^1 + \ldots + z^5 = 1 + z^1 + \ldots + z^5$

These are the first six terms of a geometric series with $a = 1$ and $r = z$

So the sum is $\sum_{k=0}^{5} z^k = \frac{z^6 - 1}{z - 1}$ **M1**

$\therefore \sum_{k=0}^{5} z^k = \frac{64-1}{(1+\mathrm{i}\sqrt{3})-1} = \frac{63}{\mathrm{i}\sqrt{3}}$ **M1**

Technique: Rationalise the denominator to obtain the answer in the required form

$= \frac{63}{\mathrm{i}\sqrt{3}} \times \frac{-\mathrm{i}\sqrt{3}}{-\mathrm{i}\sqrt{3}} = \frac{-63\mathrm{i}\sqrt{3}}{3} = -21\mathrm{i}\sqrt{3}$ **A1** **[6 Marks]**

10. a) $z^4 = 1 = \cos 0 + i\sin 0 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

Also $z^4 = (r(\cos\theta + i\sin\theta))^4 = r^4(\cos 4\theta + i\sin 4\theta)$

by de Moivre's theorem **M1**

So $1 = r^4 \therefore r = 1$ and $2k\pi = 4\theta$ so $\theta = \dfrac{k\pi}{2}$ **M1**

For $k = 0$, $\theta = 0$, so $z_1 = \cos 0 + i\sin 0 = 1$ **A1**

For $k = 1$, $\theta = \dfrac{\pi}{2}$, so $z_2 = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$ **A1**

For $k = 2$, $\theta = \dfrac{2\pi}{2} = \pi$, so $z_3 = \cos\pi + i\sin\pi = -1$ **A1**

For $k = 3$, $\theta = \dfrac{3\pi}{2}$, so $z_4 = \cos\left(\dfrac{3\pi}{2}\right) + i\sin\left(\dfrac{3\pi}{2}\right) = -i$ **A1**

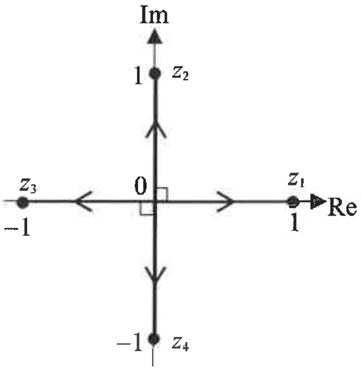
So the roots of $z^4 = 1$ are 1, i, –1, –i

Technique: Use the facts that cos θ = cos (θ + $2k\pi$) and sin θ = sin (θ + $2k\pi$) for any integer k

Technique: Obtain possible values for θ by considering different values of k

Tip: You now have four roots to the equation $z^4 = 1$, so if you continue with further values of k you will just repeat the same possible values for z

b)



A1

c) Centre is (0, 0) **B1**

Radius is 1 **B1**

d) **Show that these roots can be written as 1, ω, ω^2 and ω^3, where $1 + \omega + \omega^2 + \omega^3 = 0$, for a suitable complex number ω**

The real root is $z_1 = 1$

Let $\omega = z_2$ so $\omega = i = e^{\frac{\pi}{2}i}$ **A1**

Then $\omega^2 = \left(e^{\frac{\pi}{2}i}\right)^2 = e^{\pi i} = \cos\pi + i\sin\pi = -1 = z_3$ **A1**

and $\omega^3 = \left(e^{\frac{\pi}{2}i}\right)^3 = e^{\frac{3\pi}{2}i} = \cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2} = -i = z_4$ **A1**

So $1 + \omega + \omega^2 + \omega^3 = 1 + i - 1 - i = 0$, as required [accept $\omega = z_3$ or $\omega = z_4$] **A1** **[13 Marks]**

11. Fifth roots of unity are 1, ω, ω^2, ω^3 and ω^4, where $\omega = e^{\frac{2\pi}{5}i}$ **M1**

The vertices of the pentagon will correspond to the fifth roots of 2^5 (i.e. 32)

The root we know is 2, so the five roots are 2, 2ω, $2\omega^2$, $2\omega^3$ and $2\omega^4$ **M1**

Technique: Because the vertices of a regular pentagon with its centre at the origin represent the fifth roots of a complex number, you can use the fifth roots of unity to work out all the vertices from the vertex given in the question

So the other vertices correspond to $2\omega = 2e^{\frac{2\pi}{5}i} = 0.618033... + 1.90211...i$, i.e. (0.618, 1.90) **A1**

$2\omega^2 = 2e^{\frac{4\pi}{5}i} = -1.61803... + 1.17557...i$, i.e. (–1.62, 1.18) **A1**

$2\omega^3 = 2e^{\frac{6\pi}{5}i} = -1.61803... - 1.17557...i$, i.e. (–1.62, –1.18) **A1**

$2\omega^4 = 2e^{\frac{8\pi}{5}i} = 0.618033... - 1.90211...i$, i.e. (0.618, –1.90) **A1** **[6 Marks]**

TOTAL 57 MARKS

1. a) **Show that $\frac{1}{r}-\frac{1}{r+1}\equiv\frac{1}{r(r+1)}$**

$$\frac{1}{r}-\frac{1}{r+1}\equiv\frac{1\times(r+1)-1\times r}{r(r+1)}\equiv\frac{r+1-r}{r(r+1)}\equiv\frac{1}{r(r+1)} \quad \textbf{A1}$$

b) **By using part a) and writing out all the terms, show that $\sum_{r=1}^{5}\frac{1}{r(r+1)}=\frac{5}{6}$**

$$\sum_{r=1}^{5}\frac{1}{r(r+1)}=\sum_{r=1}^{5}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

Let $r=1: \frac{1}{1}-\frac{\cancel{1}}{\cancel{2}}$

Tip: Notice that all the terms cancel except the first and the last, so there is no need to add up every term

$r=2: \frac{\cancel{1}}{\cancel{2}}-\frac{\cancel{1}}{\cancel{3}}$

$r=3: \frac{\cancel{1}}{\cancel{3}}-\frac{\cancel{1}}{\cancel{4}}$

$r=4: \frac{\cancel{1}}{\cancel{4}}-\frac{\cancel{1}}{\cancel{5}}$

$r=5: \frac{\cancel{1}}{\cancel{5}}-\frac{1}{6}$ **M1**

So the sum is $1-\frac{1}{6}$ **M1**

$=\frac{5}{6}$ **A1** **[4 Marks]**

2. a) Let $\frac{2}{(r+1)(r+2)}\equiv\frac{A}{r+1}+\frac{B}{r+2}$ so $A(r+2)+B(r+1)\equiv 2$

$\therefore Ar+2A+Br+B=2$

Equate r coefficients: $A+B=0 \therefore A=-B$ **(1)**

Alternatively: You can let $r=-1$ to find the value of A, and let $r=-2$ to find the value of B

Equate constant terms: $2A+B=2 \therefore B=2-2A$ **(2)** **M1**

Substitute **(1)** into **(2)**:

$B=2-2(-B)=2+2B \therefore B=-2$

So, from **(1)**, $A=2$

$\therefore \frac{2}{(r+1)(r+2)}\equiv\frac{2}{r+1}-\frac{2}{r+2}$ **A1**

b) **Show that $\sum_{r=1}^{12}\frac{2}{(r+1)(r+2)}=\frac{6}{7}$, using the method of differences**

$$\sum_{r=1}^{12}\frac{2}{(r+1)(r+2)}=\sum_{r=1}^{12}\left(\frac{2}{r+1}-\frac{2}{r+2}\right)$$

Let $r=1: \frac{2}{2}-\frac{\cancel{2}}{\cancel{3}}$

Hint: Write out enough unsimplified terms of the sum to see which ones cancel and which ones remain

$r=2: \frac{\cancel{2}}{\cancel{3}}-\frac{\cancel{2}}{\cancel{4}}$

$r=3: \frac{\cancel{2}}{\cancel{4}}-\frac{\cancel{2}}{\cancel{5}}$

$\vdots$

$r=12: \frac{\cancel{2}}{\cancel{13}}-\frac{2}{14}$ **M1**

So the sum is $\frac{2}{2}-\frac{2}{14}=1-\frac{1}{7}$ **M1**

$=\frac{6}{7}$ **A1**

c) $\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)} = \sum_{r=1}^{n} \left(\frac{2}{r+1} - \frac{2}{r+2} \right)$

From part b) we know that all the terms cancel except the first and the last

Let $r = 1 : \frac{2}{2} - \cancel{\frac{2}{3}}$

$\vdots$

$r = n : \cancel{\frac{2}{n+1}} - \frac{2}{n+2}$

So the sum is $\frac{2}{2} - \frac{2}{n+2} = 1 - \frac{2}{n+2}$ **M1**

$= \frac{1 \times (n+2) - 2}{n+2} = \frac{n+2-2}{n+2} = \frac{n}{n+2}$ **A1** **[7 Marks]**

3. The Maclaurin series expansion will not be possible if:
one or more of $f(0), f'(0), f''(0), \ldots, f^{(r)}(0), \ldots$ do not have finite values **B1**
OR the function cannot be differentiated an infinite number of times **B1**
OR it does not converge **B1** **[1 Mark]**

4. a) $f(x) = \sqrt[3]{1+x} = (1+x)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}}$ **A1**

$f''(x) = -\frac{2}{3} \times \frac{1}{3}(1+x)^{-\frac{5}{3}} = -\frac{2}{9}(1+x)^{-\frac{5}{3}}$ **M1A1**

$f'''(x) = -\frac{5}{3} \times \left(-\frac{2}{9}(1+x)^{-\frac{8}{3}} \right) = \frac{10}{27}(1+x)^{-\frac{8}{3}}$ **M1A1**

b) **Show that the Maclaurin series expansion is $f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \ldots$**

Substituting $x = 0$ into the equations from part a):

$f(0) = (1+0)^{\frac{1}{3}} = 1$

$f'(0) = \frac{1}{3}(1+0)^{-\frac{2}{3}} = \frac{1}{3}$

$f''(0) = -\frac{2}{9}(1+0)^{-\frac{5}{3}} = -\frac{2}{9}$

$f'''(0) = \frac{10}{27}(1+0)^{-\frac{8}{3}} = \frac{10}{27}$ **M1**

So $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$ ◄

Technique: Substitute the values of f(0), f'(0), f''(0) and f'''(0) in the Maclaurin series expansion

$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \ldots$ **A1** **[7 Marks]**

5. a) $f(x) = e^{4x} = 1 + 4x + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + \frac{(4x)^4}{4!} + \ldots$ **M1** ◄

Technique: Replace *x* with 4*x* in the series expansion for e*ˣ* and replace *x* with 3*x* in the series expansion for ln (1 + *x*), and then simplify

$= 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \ldots$ **A1**

$g(x) = \ln(1+3x) = 3x - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} - \frac{(3x)^4}{4} + \ldots$ **M1**

$= 3x - \frac{9}{2}x^2 + 9x^3 - \frac{81}{4}x^4 + \ldots$ **A1**

b) The expansion of $f(x)$ is valid for all x **A1**

The expansion of $g(x)$ is valid for $-1 < 3x \le 1$, i.e. $-\frac{1}{3} < x \le \frac{1}{3}$ **A1** **[6 Marks]**

Technique: Adapt the normal validity condition for ln (1 + *x*) by replacing *x* with 3x

6. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$\ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \frac{(-2x)^4}{4} + \ldots$ **M1**

$= -2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \ldots$ **A1**

Technique: Use the division law of logarithms, $\log\frac{a}{b} = \log a - \log b$

So $\ln\left(\frac{1+x}{1-2x}\right) = \ln(1+x) - \ln(1-2x) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \ldots\right)$ **M1**

$= 3x + \frac{3}{2}x^2 + 3x^3 + \frac{15}{4}x^4 + \ldots \left(\text{i.e. } a = 3,\ b = \frac{15}{4}\right)$ **A1**

Technique: Take the stricter of the two conditions as the final condition for validity

The expansion of $\ln\left(\frac{1+x}{1-2x}\right)$ is valid provided both $-1 < x \le 1$ and $-1 < -2x \le 1$, i.e. $-\frac{1}{2} \le x < \frac{1}{2}$, so the expansion is only valid for $-\frac{1}{2} \le x < \frac{1}{2}$ **A1** **[5 Marks]**

7. a) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

$\therefore \cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} + \ldots$ **M1**

Technique: Replace *x* with 3*x* in the series expansion for cos *x* and replace *x* with 2*x* in the series expansion for sin *x*, and then simplify

$= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \ldots$ **A1**

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$\therefore \sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots$ **M1**

$= 2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \ldots$ **A1**

b) **Show that $2x\cos 3x - 5\sin 2x = -8x - \frac{7}{3}x^3 + \frac{65}{12}x^5 + \ldots$**

$2x\cos 3x - 5\sin 2x = 2x\left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \ldots\right) - 5\left(2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \ldots\right)$ **M1**

$= 2x - 9x^3 + \frac{27}{4}x^5 - 10x + \frac{20}{3}x^3 - \frac{4}{3}x^5 + \ldots$

$= -8x - \frac{7}{3}x^3 + \frac{65}{12}x^5 + \ldots$ **A1** **[6 Marks]**

TOTAL 36 MARKS

Solutions to Further Calculus – Test A

✓**Fundamentals** Challenge Expert

1. a) $\int \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{x} + c$ **A1**

b)
$$\int_2^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\int_2^{t} \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_2^t \quad \textbf{M1}$$

Technique: The upper limit is infinity, so replace ∞ with *t*, complete the integration, and then take the limit as *t* tends to ∞

$$= \lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{2}\right) \quad \textbf{A1}$$

$$-\frac{1}{t} \to 0 \text{ as } t\to\infty, \text{ so } \int_2^{\infty}\frac{1}{x^2}\,\mathrm{d}x \text{ converges to } \frac{1}{2} \quad \textbf{A1}$$

[4 Marks]

2. a) $\int \frac{1}{16+x^2}\,\mathrm{d}x = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + c$ **A1**

b)
$$\int \frac{1}{16+9x^2}\,\mathrm{d}x = \int \frac{1}{9\left({}^{16}\!/_{9}+x^2\right)}\,\mathrm{d}x \quad \textbf{M1}$$

Technique: Write 16 + 9x^2 in the form $k(a^2 + x^2)$ and use the standard result

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$= \frac{1}{9}\left(\frac{1}{\left({}^{4}\!/_{3}\right)}\arctan\left(\frac{x}{\left({}^{4}\!/_{3}\right)}\right)\right) + c \quad \textbf{M1}$$

$$= \frac{1}{9}\left(\frac{3}{4}\arctan\left(\frac{3x}{4}\right)\right) + c = \frac{1}{12}\arctan\left(\frac{3x}{4}\right) + c \quad \left(\text{i.e. } A = \frac{1}{12},\ B = \frac{3}{4}\right) \quad \textbf{A1}$$

[4 Marks]

3. a) Let $y = \sqrt{9-x^2}$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\times\frac{-2x}{\sqrt{9-x^2}} = \frac{-x}{\sqrt{9-x^2}}$ **A1**

b)
$$\int_0^3 \frac{2x}{\sqrt{9-x^2}}\,\mathrm{d}x = \lim_{t\to 3}\int_0^t \frac{2x}{\sqrt{9-x^2}}\,\mathrm{d}x \quad \textbf{M1}$$

$$= (-2)\times\lim_{t\to 3}\int_0^t \frac{-x}{\sqrt{9-x^2}}\,\mathrm{d}x \quad \textbf{M1}$$

Hint: Use part a)

$$= (-2)\times\lim_{t\to 3}\left[\sqrt{9-x^2}\right]_0^t = \lim_{t\to 3}\left[-2\sqrt{9-x^2}\right]_0^t \quad \textbf{M1}$$

$$= \lim_{t\to 3}\left(-2\sqrt{9-t^2} - \left(-2\sqrt{9}\right)\right)$$

$$= \lim_{t\to 3}\left(-2\sqrt{9-t^2} + 6\right) \quad \textbf{M1}$$

$$= 6 \ \left(\text{since as } t\to 3,\ \sqrt{9-t^2} \to \sqrt{9-9} = 0\right) \quad \textbf{A1}$$

[6 Marks]

4. **Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$**

Let $y = \arcsin x$, so $x = \sin y$ **M1**

$\frac{\mathrm{d}x}{\mathrm{d}y} = \cos y$ **M1**

Technique: Differentiate *x* with respect to *y*, and then use the relation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$

Alternatively: You could use implicit differentiation to differentiate $x = \sin y$ with respect to *x*

So $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y}$ **M1**

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \textbf{A1}$$

[4 Marks]

Technique: Use the identity $\cos^2 y + \sin^2 y \equiv 1$ and the fact that $\cos y$ is positive in the range of arcsin to rewrite the denominator in terms of *x*

5. $\int_1^4 3x \, \mathrm{d}x = \left[\frac{3}{2}x^2\right]_1^4$ **M1**

$= \frac{3}{2}\times 4^2 - \frac{3}{2}\times 1^2$ **M1**

$= \frac{45}{2}$ **A1**

The mean value of $3x$ on $[1, \ 4]$ is $\frac{1}{4-1}\int_1^4 3x \, \mathrm{d}x$ **M1**

$= \frac{1}{3}\left(\frac{45}{2}\right) = \frac{15}{2}$ **A1**

Technique: The mean value of f(x) over the interval [a, b] is $\frac{1}{b-a}\int_a^b \text{f}(x)\,\text{d}x$

[5 Marks]

6. $x = \frac{1}{2}\cos\theta$, so $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{1}{2}\sin\theta \therefore \mathrm{d}x = -\frac{1}{2}\sin\theta \, \mathrm{d}\theta$

$\therefore \int \frac{-1}{\sqrt{1-4x^2}} \, \mathrm{d}x = \int \frac{-1}{\sqrt{1-4\left(\frac{1}{2}\cos\theta\right)^2}} \times \left(-\frac{1}{2}\sin\theta\right) \mathrm{d}\theta$ **M1**

Technique: Substitute for x and use the identity $\cos^2\theta + \sin^2\theta \equiv 1$ to simplify

$= \int \frac{\frac{1}{2}\sin\theta}{\sqrt{1-4\times\frac{1}{4}\cos^2\theta}} \, \mathrm{d}\theta = \int \frac{\frac{1}{2}\sin\theta}{\sqrt{1-\cos^2\theta}} \, \mathrm{d}\theta$ **M1**

$= \int \frac{\frac{1}{2}\sin\theta}{\sqrt{\sin^2\theta}} \, \mathrm{d}\theta = \int \frac{\frac{1}{2}\sin\theta}{\sin\theta} \, \mathrm{d}\theta = \int \frac{1}{2} \, \mathrm{d}\theta = \frac{1}{2}\theta + c$ **M1**

Since $x = \frac{1}{2}\cos\theta$, $2x = \cos\theta$ so $\theta = \arccos(2x)$

Hint: Remember to rewrite the answer in terms of x at the end

$\therefore \int \frac{-1}{\sqrt{1-4x^2}} \, \mathrm{d}x = \frac{1}{2}\theta + c = \frac{1}{2}\arccos(2x) + c$, i.e. $A = \frac{1}{2}$ and $B = 2$ **A1**

[4 Marks]

7. a) Let $\frac{2x^2-6x+17}{(2x-3)(x^2+4)} \equiv \frac{A}{2x-3} + \frac{B}{x^2+4}$ so $A(x^2+4) + B(2x-3) \equiv 2x^2-6x+17$

$\therefore Ax^2 + 4A + 2Bx - 3B = 2x^2 - 6x + 17$

Equate x^2 coefficients: $A = 2$ **A1**

Equate x coefficients: $2B = -6 \therefore B = -3$ **A1**

So $\frac{2x^2-6x+17}{(2x-3)(x^2+4)} = \frac{2}{2x-3} - \frac{3}{x^2+4}$ (i.e. $A = 2$, $B = -3$) **A1**

Tip: Check that the constant terms are also consistent with these values; $4A - 3B = 4 \times 2 - 3 \times (-3) = 17$, which is correct

b) $\int \frac{2x^2-6x+17}{(2x-3)(x^2+4)} \, \mathrm{d}x = \int \left(\frac{2}{2x-3} - \frac{3}{x^2+4}\right) \mathrm{d}x = \int \frac{2}{2x-3} \, \mathrm{d}x - \int \frac{3}{x^2+4} \, \mathrm{d}x$ **M1**

$= \ln|2x-3| - \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c$ **M1**

i.e. $C = 1$ and $D = -\frac{3}{2}$ **A1**

[6 Marks]

8. $y = \sqrt{\sin 2x}$

$V = \pi\int_0^{\frac{\pi}{4}} \left(\sqrt{\sin 2x}\right)^2 \mathrm{d}x$

Technique: If the curve with equation $y = \text{f}(x)$ is rotated 2π radians about the x-axis between $x = a$ and $x = b$, then the volume of the solid generated is given by $V = \pi\int_a^b y^2 \, \text{d}x = \pi\int_a^b (\text{f}(x))^2 \, \text{d}x$

$= \pi\int_0^{\frac{\pi}{4}} \sin 2x \, \mathrm{d}x$ **M1**

$= \pi\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$ **M1**

$= \pi\left(-\frac{1}{2}\cos\left(2\times\frac{\pi}{4}\right) + \frac{1}{2}\cos(2\times 0)\right)$ **M1**

$= \pi\left(0 + \frac{1}{2}\right) = \frac{\pi}{2}$ **A1**

[4 Marks]

9. $x = 12\cos\theta$ so $x^2 = 144\cos^2\theta$ **M1**

$y = 10\sin\theta$ so $\dfrac{dy}{d\theta} = 10\cos\theta$ **M1**

So $V = \pi\int_0^{\frac{\pi}{2}} 144\cos^2\theta \times 10\cos\theta \; d\theta = \pi\int_0^{\frac{\pi}{2}} 1440\cos^3\theta \; d\theta$ **M1**

> **Technique:** If the curve with parametric equations $x = f(\theta)$, $y = g(\theta)$ is rotated 2π radians about the y-axis between $\theta = a$ and $\theta = b$, then the volume of the solid generated is given by $V = \pi\int_a^b x^2 \dfrac{dy}{d\theta} d\theta$

We are given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, so $\cos^3\theta \equiv \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$$\therefore V = \pi\int_0^{\frac{\pi}{2}} 1440\left(\frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\right) d\theta = \pi\int_0^{\frac{\pi}{2}} \left(360\cos 3\theta + 1080\cos\theta\right) d\theta \quad \textbf{M1}$$

$$= \pi\left[120\sin 3\theta + 1080\sin\theta\right]_0^{\frac{\pi}{2}} \quad \textbf{M1}$$

$$= \pi\left(\left(120\sin\frac{3\pi}{2} + 1080\sin\frac{\pi}{2}\right) - \left(120\sin 0 + 1080\sin 0\right)\right) \quad \textbf{M1}$$

$$= \pi\left(-120 + 1080 - 0 - 0\right) = 960\pi \text{ cm}^3 \quad \textbf{A1} \quad \textbf{[7 Marks]}$$

TOTAL 44 MARKS

Solutions to Polar Coordinates – Test A

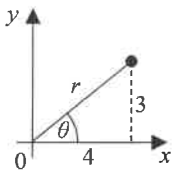
✓**Fundamentals** Challenge Expert

1. a) $(x, y) = (4, 3)$

$$r = \sqrt{4^2+3^2} = \sqrt{25} = 5$$

$$\theta = \arctan\left(\frac{3}{4}\right) = 0.643501\ldots \quad \textbf{M1}$$

So $(r, \theta) = (5, 0.644)$ (3 s.f.) **A1**



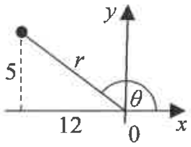
Tip: If P is in the first quadrant and has Cartesian coordinates (x, y) then it has polar coordinates (r, θ) where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. If P is not in the first quadrant, then a diagram helps you determine how to find θ.

b) $(x, y) = (-12, 5)$

$$r = \sqrt{(-12)^2+5^2} = \sqrt{169} = 13$$

$$\theta = \pi - \arctan\left(\frac{5}{12}\right) = 2.74680\ldots \quad \textbf{M1}$$

So $(r, \theta) = (13, 2.75)$ (3 s.f.) **A1**

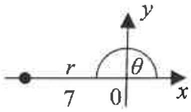


c) $(x, y) = (-7, 0)$

$$r = \sqrt{(-7)^2+0^2} = \sqrt{49} = 7$$

$$\theta = \pi - \arctan\left(\frac{0}{7}\right) = \pi \quad \textbf{M1}$$

So $(r, \theta) = (7, \pi)$ **A1**

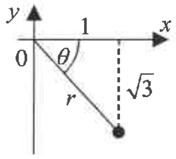


d) $(x, y) = \left(1, -\sqrt{3}\right)$

$$r = \sqrt{1^2+\left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

$$\theta = -\arctan\left(\frac{\sqrt{3}}{1}\right) = -\frac{\pi}{3} \quad \textbf{M1}$$

So $(r,\theta) = \left(2, -\frac{\pi}{3}\right)$ **A1**



[8 Marks]

2. a) $(r, \theta) = (5, \pi)$

$x = r\cos\theta = 5\cos\pi = 5\times(-1) = -5$

$y = r\sin\theta = 5\sin\pi = 5\times 0 = 0$ **M1**

So $(x, y) = (-5, 0)$ **A1**

Tip: If P has polar coordinates (r, θ), then it has Cartesian coordinates (x, y) where $x = r\cos\theta$ and $y = r\sin\theta$

b) $(r,\theta) = \left(5, \frac{3\pi}{2}\right)$

$$x = r\cos\theta = 5\cos\frac{3\pi}{2} = 5\times 0 = 0$$

$$y = r\sin\theta = 5\sin\frac{3\pi}{2} = 5\times(-1) = -5 \quad \textbf{M1}$$

So $(x, y) = (0, -5)$ **A1**

c) $(r,\theta) = \left(2, \frac{7\pi}{4}\right)$

$$x = r\cos\theta = 2\cos\frac{7\pi}{4} = 2\times\frac{\sqrt{2}}{2} = \sqrt{2}$$

$$y = r\sin\theta = 2\sin\frac{7\pi}{4} = 2\times\left(-\frac{\sqrt{2}}{2}\right) = -\sqrt{2} \quad \textbf{M1}$$

So $(x,y) = \left(\sqrt{2}, -\sqrt{2}\right)$ **A1**

d) $(r,\theta) = \left(4, \frac{2\pi}{3}\right)$

$$x = r\cos\theta = 4\cos\frac{2\pi}{3} = 4\times\left(-\frac{1}{2}\right) = -2$$

$$y = r\sin\theta = 4\sin\frac{2\pi}{3} = 4\times\frac{\sqrt{3}}{2} = 2\sqrt{3} \quad \textbf{M1}$$

So $(x,y) = \left(-2, 2\sqrt{3}\right)$ **A1**

[8 Marks]

3. a) $r = 5$

$\therefore\ r^2 = 25$

$\therefore\ x^2 + y^2 = 25$ **M1A1**

Tip: To convert a polar equation into a Cartesian one, manipulate it until it is composed of terms involving r^2, $r\cos\theta$ and/or $r\sin\theta$. Then use the substitutions $r^2 = x^2 + y^2$, $r\cos\theta = x$, and $r\sin\theta = y$.

b) $r\cos\theta = 2$

$\therefore\ x = 2$ **M1A1**

c) $r = 3\operatorname{cosec}\theta$

$\therefore\ r\sin\theta = 3$

$\therefore\ y = 3$ **M1A1**

d) $r^2 = \cos\theta\sin\theta$

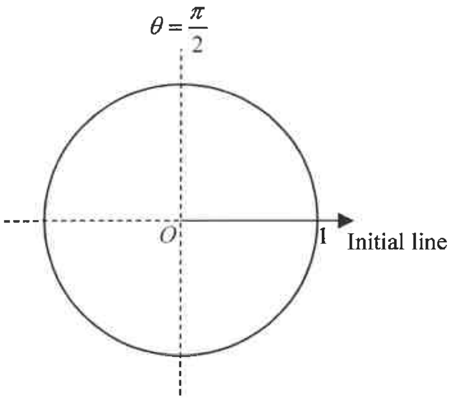
$\therefore\ r^4 = r^2\cos\theta\sin\theta$

Technique: Multiply both sides by r^2, and then rearrange to get terms of the desired form

$\therefore\ \left(r^2\right)^2 = \left(r\cos\theta\right)\left(r\sin\theta\right)$

$\therefore\ \left(x^2 + y^2\right)^2 = xy$ **M1A1** **[8 Marks]**

4. a) $r = 1$

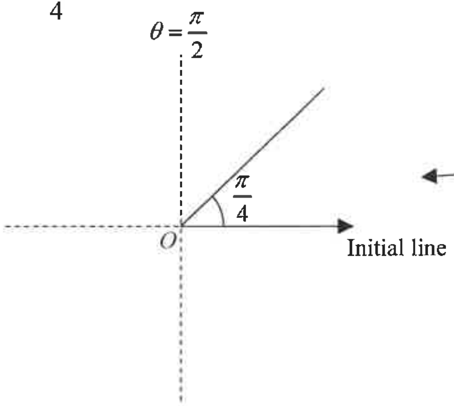


Tip: If $a > 0$ then the curve with equation $r = a$ is just a circle with radius a and centre O

For a circle **A1**

For a centre at the origin **A1**

b) $\theta = \frac{\pi}{4}$

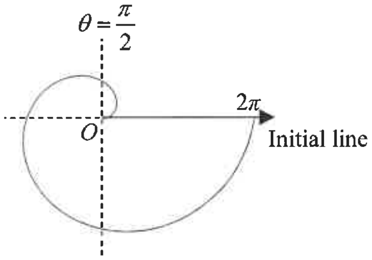


Tip: The equation $\theta = \alpha$ gives a half-line that starts at the origin and that makes an angle α with the initial line

For a half-line starting at the origin **A1**

For a line that makes an angle of $\frac{\pi}{4}$ with the initial line **A1**

c) $r = \theta$



For a spiral that increases in radius as θ increases **A1**

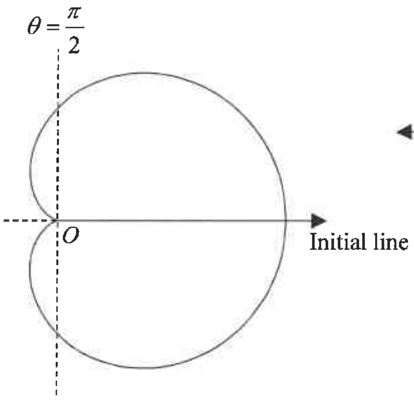
For a curve starting at the origin **A1** **[6 Marks]**

5. a)

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
r	2	1.92	**1.71**	1.38	**1**	0.62	0.29	**0.08**	**0**

A1

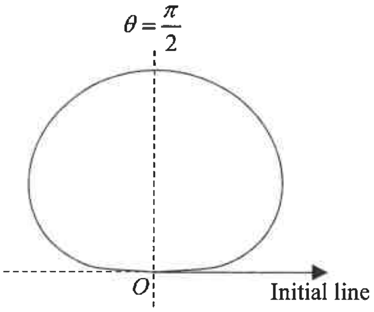
b) $\cos(-\theta) \equiv \cos\theta$, so the curve will be symmetric about the initial line



Tip: Use the values for r you found in part a) to help draw the upper half of the curve, and then use its symmetry to complete the lower half

For cardioid shape above the initial line for $0 \le \theta \le \pi$ **A1**
For correct shape for all θ **A1** **[3 Marks]**

6.



Tip: There are several polar curves you are expected to be able to draw, including this one. You can always make a table like the one in 5a) to give you an idea of the curve's shape.

For egg-shaped curve **A1**
For correct orientation **A1** **[2 Marks]**

7. $r = \pi - \theta$, so area $= \frac{1}{2}\int_0^{\pi}(\pi-\theta)^2\,\mathrm{d}\theta$ **M1**

$$= \frac{1}{2}\int_0^{\pi}\left(\pi^2 - 2\pi\theta + \theta^2\right)\mathrm{d}\theta$$

$$= \frac{1}{2}\left[\pi^2\theta - \pi\theta^2 + \frac{\theta^3}{3}\right]_0^{\pi}$$ **M1**

$$= \frac{1}{2}\left(\left(\pi^2\times\pi - \pi\times\pi^2 + \frac{\pi^3}{3}\right) - \left(\pi^2\times 0 - \pi\times 0 + \frac{0}{3}\right)\right)$$

$$= \frac{\pi^3}{6}$$ **A1** **[3 Marks]**

Tip: The area enclosed by the curve $r = \mathrm{f}(\theta)$ between $\theta = \alpha$ and $\theta = \beta$ is given by the formula $\frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$, or $\frac{1}{2}\int_{\alpha}^{\beta}(\mathrm{f}(\theta))^2\,d\theta$

8. $r = \mathrm{e}^{\theta}$
$y = r\sin\theta = \mathrm{e}^{\theta}\sin\theta$
$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{e}^{\theta}\sin\theta + \mathrm{e}^{\theta}\cos\theta = \mathrm{e}^{\theta}(\cos\theta + \sin\theta)$ **M1**

Tip: Use the product rule

So $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$ when $\mathrm{e}^{\theta} = 0$ or when $\cos\theta + \sin\theta = 0$ **M1**

Tip: The curve is parallel to the initial line when $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$

e^{θ} is never zero, so we are left with $\cos\theta + \sin\theta = 0$. Since $\cos\theta$ and $\sin\theta$ are never simultaneously zero, we can divide by $\cos\theta$ to get:
$1 + \tan\theta = 0$
$\therefore \theta = \arctan(-1)$ **M1**

We want the solution in the range $0 \le \theta \le \pi$, which is $\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} = 2.35619... = 2.36$ (3 s.f.)

and so $r = \mathrm{e}^{\frac{3\pi}{4}} = 10.5507... = 10.6$ (3 s.f.)
So P has polar coordinates (10.6, 2.36) (3 s.f.) **A1** **[4 Marks]**

9. $r = 2\sin\theta$

$x = r\cos\theta = 2\sin\theta\cos\theta = \sin 2\theta$

Hint: Using the double angle formula for sine here means you do not have to use the product rule to differentiate *x*

$\dfrac{dx}{d\theta} = 2\cos 2\theta$ **M1**

Tip: The curve is perpendicular to the initial line when $\dfrac{dx}{d\theta} = 0$

So $\dfrac{dx}{d\theta} = 0$ when $2\cos 2\theta = 0$ **M1**

$\therefore 2\theta = \arccos(0)$ **M1**

We want solutions in the range $0 \le \theta \le \pi$, which means $0 \le 2\theta \le 2\pi$

So $2\theta = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

$\therefore \theta = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$ **A1**

When $\theta = \dfrac{\pi}{4}$, $r = 2\sin\dfrac{\pi}{4} = 2\times\dfrac{\sqrt{2}}{2} = \sqrt{2}$

When $\theta = \dfrac{3\pi}{4}$, $r = 2\sin\dfrac{3\pi}{4} = 2\times\dfrac{\sqrt{2}}{2} = \sqrt{2}$

So the points are $\left(\sqrt{2}, \dfrac{\pi}{4}\right)$ and $\left(\sqrt{2}, \dfrac{3\pi}{4}\right)$ **A1** **[5 Marks]**

TOTAL 47 MARKS

Solutions to Hyperbolic Functions – Test A

✓ **Fundamentals** Challenge Expert

1\. a) $\cosh 2 = \dfrac{e^2 + e^{-2}}{2}$ **M1**

$= 3.76219\ldots = 3.76 \text{ (2 d.p.)}$ **A1**

b) $\sinh \dfrac{1}{3} = \dfrac{e^{\frac{1}{3}} - e^{-\frac{1}{3}}}{2}$ **M1**

$= 0.339540\ldots = 0.34 \text{ (2 d.p.)}$ **A1**

c) $\tanh(-1) = \dfrac{e^{2(-1)} - 1}{e^{2(-1)} + 1} = \dfrac{e^{-2} - 1}{e^{-2} + 1}$ **M1**

$= -0.761594\ldots = -0.76 \text{ (2 d.p.)}$ **A1** **[6 Marks]**

Alternatively: You can use the definitions of sinh x and cosh x to work out $\tanh(-1) = \dfrac{\sinh(-1)}{\cosh(-1)}$

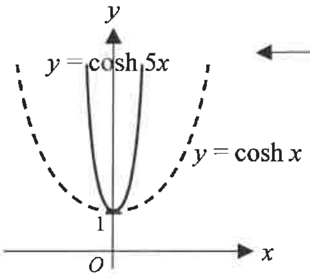
2\. $\sinh(\ln 3) = \dfrac{e^{\ln 3} - e^{-\ln 3}}{2}$ **M1**

$= \dfrac{3 - \frac{1}{3}}{2}$ **M1**

$= \dfrac{\frac{8}{3}}{2} = \dfrac{4}{3}$ **A1** **[3 Marks]**

Hint: Use the power law of logarithms, $a\log b = \log b^a$, to convert $-\ln 3$ to $\ln\dfrac{1}{3}$

3\. a)

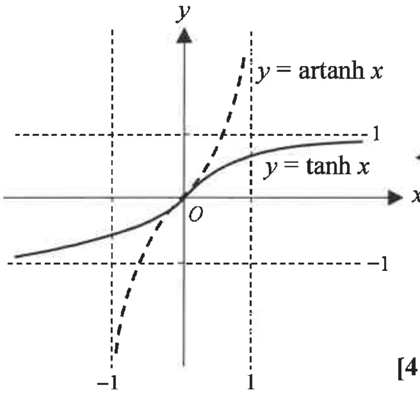


A1A1

Hint: $y = \cosh 5x$ is a stretch of $y = \cosh x$ by scale factor $1/5$ parallel to the x-direction

b) $f(x) \geq 1$ **A1** **[3 Marks]**

4\.



A1 correct shape for $y = \tanh x$

A1 horizontal asymptotes at $y = -1$ and $y = 1$

A1 correct shape for $y = \text{artanh}\, x$

A1 vertical asymptotes at $x = -1$ and $x = 1$

[4 Marks]

Tip: Remember that the graph of an inverse function is a reflection of the graph of the corresponding function in the line $y = x$

5\. a) $\text{arcosh}\, 4 = \ln\left(4 + \sqrt{4^2 - 1}\right)$ **M1**

$= \ln\left(4 + \sqrt{15}\right)$ **A1**

Tip: The logarithmic forms for inverse hyperbolic functions can be found in your formula book in the 'Hyperbolic functions' section

b) $\text{arsinh}\sqrt{3} = \ln\left(\sqrt{3} + \sqrt{\left(\sqrt{3}\right)^2 + 1}\right)$ **M1**

$= \ln\left(\sqrt{3} + \sqrt{4}\right) = \ln\left(\sqrt{3} + 2\right)$ **A1**

c) $\text{artanh}\, \dfrac{1}{4} = \dfrac{1}{2}\ln\left(\dfrac{1 + \frac{1}{4}}{1 - \frac{1}{4}}\right)$ **M1**

$= \dfrac{1}{2}\ln\left(\dfrac{\frac{5}{4}}{\frac{3}{4}}\right) = \dfrac{1}{2}\ln\left(\dfrac{5}{3}\right) = \ln\sqrt{\dfrac{5}{3}}$ **A1** **[6 Marks]**

6. a) $\int \frac{7}{\sqrt{x^2+1}}\,\mathrm{d}x = 7\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x = 7\,\mathrm{arsinh}\,x + c$ **A1**

Tip: Be careful – the inverse trigonometric and inverse hyperbolic standard results for integration look very similar, so make sure you get the right one!

b) $\int \frac{3}{\sqrt{x^2-1}}\,\mathrm{d}x = 3\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x = 3\,\mathrm{arcosh}\,x + c$ **A1**

[2 Marks]

7. a) $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh 4x) = 4\sinh 4x$ **A1**

Technique: Use the chain rule for parts a) and b), and the product and chain rule for parts c) and d)

b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}\tanh 3x\right) = \frac{1}{3}\times 3\times \mathrm{sech}^2\,3x$

$= \mathrm{sech}^2\,3x$ **A1**

c) $\frac{\mathrm{d}}{\mathrm{d}x}(x\sinh 2x) = 1\times\sinh 2x + x\times 2\cosh 2x$ **M1**

$= \sinh 2x + 2x\cosh 2x$ **A1**

d) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln 2x\cosh 2x) = \frac{2}{2x}\times\cosh 2x + \ln 2x\times 2\sinh 2x$ **M1**

$= \frac{\cosh 2x}{x} + 2\ln 2x\sinh 2x$ **A1**

[6 Marks]

8. **Show that $\tanh 2A \equiv \frac{2\tanh A}{1+\tanh^2 A}$**

$LHS = \tanh 2A \equiv \frac{\sinh 2A}{\cosh 2A}$ **M1**

$\equiv \frac{2\sinh A\cosh A}{\cosh^2 A + \sinh^2 A}$ **M1**

Technique: Divide the numerator and the denominator by cosh² A

$\equiv \frac{2\frac{\sinh A}{\cosh A}}{1+\frac{\sinh^2 A}{\cosh^2 A}} \equiv \frac{2\tanh A}{1+\tanh^2 A} = RHS$ **A1**

[3 Marks]

9. $\cosh x + 2\sinh x = 1$

$\frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} + 2\left(\frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}\right) = 1$

$\frac{1}{2}\mathrm{e}^x + \frac{1}{2}\mathrm{e}^{-x} + \mathrm{e}^x - \mathrm{e}^{-x} - 1 = 0$ **M1**

$\frac{3}{2}\mathrm{e}^x - 1 - \frac{1}{2}\mathrm{e}^{-x} = 0$ **M1**

$3\mathrm{e}^{2x} - 2\mathrm{e}^x - 1 = 0$

Technique: Rewrite the equation in exponential form then multiply by 2eˣ to get a quadratic equation in eˣ, and factorise to solve

$(3\mathrm{e}^x + 1)(\mathrm{e}^x - 1) = 0$ **M1**

So $\mathrm{e}^x = -\frac{1}{3}$ or $\mathrm{e}^x = 1$ **M1**

There are no real values of x for which $\mathrm{e}^x = -\frac{1}{3}$. Hence $\mathrm{e}^x = 1$, so $x = \ln 1 = 0$. **A1A1** **[6 Marks]**

10. **Prove that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = ny$**

$y = 2\cosh 4x + 3\sinh 4x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\times 4\sinh 4x + 3\times 4\cosh 4x = 8\sinh 4x + 12\cosh 4x$ **M1**

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 8\times 4\cosh 4x + 12\times 4\sinh 4x = 32\cosh 4x + 48\sinh 4x$ **M1**

$= 16(2\cosh 4x + 3\sinh 4x)$ **M1**

$= 16y$, so $n = 16$ **A1**

[4 Marks]

11. Since $\frac{\mathrm{d}}{\mathrm{d}x}(\cosh 2x) = 2\sinh 2x$, to find $\int \cosh^3 2x \sinh 2x \, \mathrm{d}x$, try $y = \cosh^4 2x$:

Then $\frac{\mathrm{d}y}{\mathrm{d}x} = 4\cosh^3 2x \times 2\sinh 2x = 8\cosh^3 2x \sinh 2x$ **M1**

So $\int \cosh^3 2x \sinh 2x \, \mathrm{d}x = \frac{1}{8}\cosh^4 2x + c$ **A1** **[2 Marks]**

Technique: Use the reverse chain rule to integrate a function of the form $kf'(x)(f(x))^n$ by trying $(f(x))^{n+1}$, where $n \neq -1$

Alternatively: You could also use the substitution $u = \cosh 2x$

12. $\int e^{3x} \cosh x \, \mathrm{d}x = \int e^{3x}\left(\frac{e^x + e^{-x}}{2}\right) \mathrm{d}x$ **M1**

$$= \frac{1}{2}\int \left(e^{4x} + e^{2x}\right) \mathrm{d}x \quad \textbf{M1}$$

$$= \frac{1}{2}\left(\frac{1}{4}e^{4x} + \frac{1}{2}e^{2x}\right) + c = \frac{1}{8}\left(e^{4x} + 2e^{2x}\right) + c \quad \textbf{A1}$$

[3 Marks]

13. From the formula book, $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$ and $\cosh^2 x - \sinh^2 x \equiv 1$

So $\cosh 2x \equiv \cosh^2 x + \left(\cosh^2 x - 1\right) \equiv 2\cosh^2 x - 1$ **M1**

$\therefore \cosh^2 x \equiv \frac{1}{2}(1 + \cosh 2x)$

So $\int \cosh^2 x \, \mathrm{d}x = \frac{1}{2}\int (1 + \cosh 2x) \, \mathrm{d}x$ **M1**

$$= \frac{1}{2}\left(x + \frac{1}{2}\sinh 2x\right) + c = \frac{1}{2}x + \frac{1}{4}\sinh 2x + c \quad \textbf{A1}$$

[3 Marks]

14. $x = 5\sinh u$ so $\frac{\mathrm{d}x}{\mathrm{d}u} = 5\cosh u \therefore \mathrm{d}x = 5\cosh u \, \mathrm{d}u$

$$\therefore \int \frac{1}{\sqrt{x^2 + 25}} \mathrm{d}x = \int \frac{1}{\sqrt{(5\sinh u)^2 + 25}} \times (5\cosh u \, \mathrm{d}u) \quad \textbf{M1}$$

Technique: Substitute for x and simplify

$$= \int \frac{5\cosh u}{\sqrt{25\sinh^2 u + 25}} \mathrm{d}u = \int \frac{5\cosh u}{\sqrt{25\left(\sinh^2 u + 1\right)}} \mathrm{d}u \quad \textbf{M1}$$

Technique: Use the identity $\cosh^2 u - \sinh^2 u \equiv 1$ to rewrite the denominator in terms of $\cosh u$

$$= \int \frac{5\cosh u}{\sqrt{25\cosh^2 u}} \mathrm{d}u = \int \frac{5\cosh u}{5\cosh u} \mathrm{d}u = \int 1 \, \mathrm{d}u = u + c \quad \textbf{M1}$$

Since $x = 5\sinh u$, $\frac{x}{5} = \sinh u$ so $u = \operatorname{arsinh}\left(\frac{x}{5}\right)$

Technique: Rewrite the answer in terms of x

$$\therefore \int \frac{1}{\sqrt{x^2 + 25}} \mathrm{d}x = u + c = \operatorname{arsinh}\left(\frac{x}{5}\right) + c \quad \textbf{A1}$$

[4 Marks]

TOTAL 55 MARKS

Solutions to Differential Equations – Test A

✓ **Fundamentals** Challenge Expert

1. a) $\frac{dy}{dx} + 2xy = 1$

Integrating factor $= e^{\int 2x\,dx}$ **M1**

$= e^{x^2}$ **A1**

Technique: The integrating factor for the equation $\frac{dy}{dx} + P(x)y = Q(x)$ is $e^{\int P(x)dx}$

b) $\frac{dy}{dx} + y = x$

Integrating factor $= e^{\int 1\,dx}$ **M1**

$= e^x$ **A1** **[4 Marks]**

2. $\frac{dy}{dx} + \frac{y}{x} = 3x$

We are told in the question that the integrating factor is x. So, multiplying through by this:

$x\frac{dy}{dx} + y = 3x^2$ **M1**

$\therefore \frac{d}{dx}(xy) = 3x^2$

$\therefore xy = \int 3x^2\,dx$ **M1**

$= x^3 + c$ **A1**

Technique: Multiply through by the integrating factor g(x). The left-hand side is then the derivative of g(x)y with respect to x.

Technique: Integrate both sides with respect to x so that the left-hand side becomes g(x)y

So the general solution is $y = \frac{x^3 + c}{x}$ **A1** **[4 Marks]**

3. a) $y'' - 3y' + 2y = 0$

The auxiliary equation is $m^2 - 3m + 2 = 0$ **M1**

$\therefore (m-1)(m-2) = 0$

So $m = 1$ or $m = 2$ **A1**

So the general solution is $y = Ae^x + Be^{2x}$ **A1**

Tip: The auxiliary equation for $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$

Tip: If the auxiliary equation has distinct real roots α and β, then the general solution to the differential equation is $y = Ae^{\alpha x} + Be^{\beta x}$

b) $y'' - 2y' + y = 0$

The auxiliary equation is $m^2 - 2m + 1 = 0$ **M1**

$\therefore (m-1)^2 = 0$

So $m = 1$ is a repeated root **A1**

So the general solution is $y = (A + Bx)e^x$ **A1**

Tip: If the auxiliary equation has a repeated real root α, then the general solution to the differential equation is $y = (A + Bx)e^{\alpha x}$

c) $y'' - 2y' + 2y = 0$

The auxiliary equation is $m^2 - 2m + 2 = 0$ **M1**

Completing the square, this is $(m-1)^2 + 1 = 0$, so $m = 1 \pm i$ **A1**

So the general solution is $y = e^x(A\cos x + B\sin x)$ **A1**

Tip: If the auxiliary equation has complex conjugate roots $p \pm qi$, then the general solution to the differential equation is $y = e^{px}(A\cos qx + B\sin qx)$

[9 Marks]

4. $y'' - 4y' - 5y = 3e^{-x}$

a) The complementary function satisfies $y'' - 4y' - 5y = 0$

The auxiliary equation is $m^2 - 4m - 5 = 0$ **M1**

$\therefore (m+1)(m-5) = 0$

So $m = -1$ or $m = 5$ **A1**

So the complementary function is $y = Ae^{-x} + Be^{5x}$ **A1**

Tip: The complementary function for a non-homogeneous second-order differential equation is a solution to the homogeneous equation formed by setting the right-hand side equal to zero

b) e^{-x} is part of the complementary function, so is unsuitable for use in the particular integral **B1** **[4 Marks]**

5. Velocity is the derivative of displacement with respect to time, so $\frac{dx}{dt} = 2t + \cos t$ **M1**

$\therefore x = \int (2t + \cos t)\,dt$

$= t^2 + \sin t + c$ **A1**

We are told in the question that $x = 0$ when $t = \pi$, so $0 = \pi^2 + \sin\pi + c = \pi^2 + c$, so $c = -\pi^2$

and so $x = t^2 + \sin t - \pi^2$ **A1**

When $t = 2\pi$ s, $x = (2\pi)^2 + \sin 2\pi - \pi^2 = 3\pi^2$ m **A1** **[4 Marks]**

6. a) **Show that the particle's velocity, *v*, satisfies $v\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1}{4}x$**

Velocity is the derivative of displacement with respect to time, $\dot{x}$

> **Tip:** The dot notation represents differentiation with respect to time. If x represents displacement, then $\dot{x}$ is velocity and $\ddot{x}$ is acceleration.

Also, $\ddot{x}=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}v}{\mathrm{d}x}\times v=v\frac{\mathrm{d}v}{\mathrm{d}x}$ **M1**

and so $\ddot{x}=-\frac{1}{4}x$ becomes $v\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1}{4}x$ **A1**

b) From part a), $v\frac{\mathrm{d}v}{\mathrm{d}x}=-\frac{1}{4}x$

$\therefore \int v\,\mathrm{d}v=\int -\frac{1}{4}x\,\mathrm{d}x$

$\therefore \frac{1}{2}v^2=-\frac{1}{8}x^2+c$ **A1**

> **Tip:** The question gives you values for x and v at a certain time. Substitute these into the general solution to determine the constant of integration, c.

We are told in the question that $v=\frac{1}{2}$ and $x=0$ when $t=0$, so $\frac{1}{2}\times\left(\frac{1}{2}\right)^2=-\frac{1}{8}\times0+c$, so $c=\frac{1}{8}$ **A1**

So $\frac{1}{2}v^2=-\frac{1}{8}x^2+\frac{1}{8}$

$\therefore 4v^2=1-x^2$ **A1**

(So $k=4$) **[5 Marks]**

7. **Verify that $x=\sin t+2\cos t$ is a particular solution to the equation $\ddot{x}+4\dot{x}+9x=20\cos t$**

$x=\sin t+2\cos t$

$\therefore \dot{x}=\cos t-2\sin t$ **A1**

$\therefore \ddot{x}=-\sin t-2\cos t$ **A1**

So $LHS=\ddot{x}+4\dot{x}+9x$

$=-\sin t-2\cos t+4(\cos t-2\sin t)+9(\sin t+2\cos t)$ **M1**

$=(-1-8+9)\sin t+(-2+4+18)\cos t$

$=20\cos t=RHS$ **A1** **[4 Marks]**

8. $\frac{\mathrm{d}x}{\mathrm{d}t}=2y$

$\frac{\mathrm{d}y}{\mathrm{d}t}=3x+y$

a) **Show that $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{\mathrm{d}x}{\mathrm{d}t}-6x=0$**

> **Technique:** You can use the first equation to write y in terms of the derivative of x, and then differentiate this with respect to t to find $\frac{\mathrm{d}y}{\mathrm{d}t}$ in terms of x and its derivatives. You can then substitute these expressions into the second equation.

From the first differential equation, $y=\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$ **A1**

So $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ **A1**

Substituting these into the second differential equation gives:

$\frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=3x+\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}$ **M1**

$\therefore \frac{1}{2}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}-3x=0$

$\therefore \frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{\mathrm{d}x}{\mathrm{d}t}-6x=0$ **A1**

b) The auxiliary equation of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{\mathrm{d}x}{\mathrm{d}t}-6x=0$ is $m^2-m-6=0$ **M1**

$\therefore (m+2)(m-3)=0$

So $m=-2$ or $m=3$ **A1**

So the general solution is $x=A\mathrm{e}^{-2t}+B\mathrm{e}^{3t}$ **A1** **[7 Marks]**

TOTAL 41 MARKS