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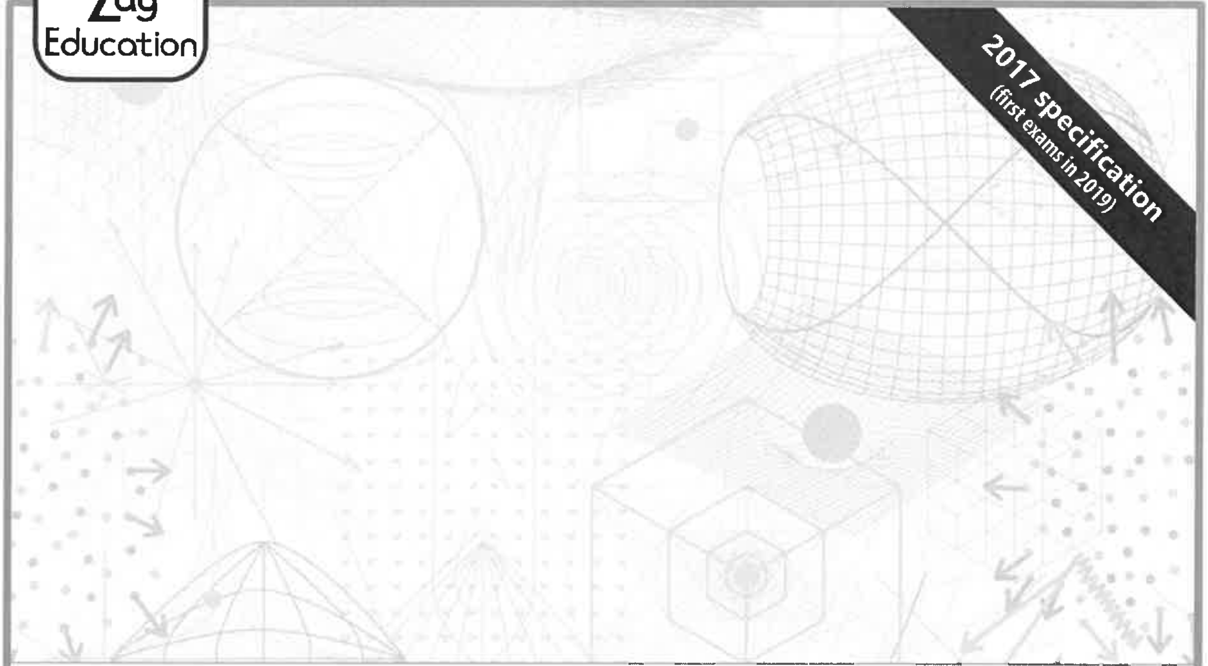

Topic Tests: Fundamentals Tests – Set A

A Level Edexcel Further Mathematics: Core Pure Mathematics: Part 1#

#Every topic of AS (8FM0) Core Pure Mathematics

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Contents

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This resource is cross-referenced to the following textbook: the Pearson Education textbook *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* by Greg Attwood, Jack Barraclough, Ian Bettison, Lee Cope, Charles Garnet Cox, Daniel Goldberg, Alistair Macpherson, Bronwen Moran, Su Nicholson, Laurence Pateman, Joe Petran, Keith Pledger, Harry Smith, Geoff Staley and Dave Wilkins (ISBN 978-1292183336). ZigZag Education is not affiliated with Pearson Education in any way nor is this publication authorised by, associated with, sponsored by or endorsed by Pearson Education unless explicitly stated on the front cover of this publication.

Teacher's Introduction

Content

This pack contains 6 fundamentals level topic tests for A Level Edexcel Further Mathematics: Core Pure Mathematics.

Each test comes with fully worked solutions, containing tips, hints and technique boxes to help students on a particular question. Answers should be given to three significant figures unless specified in the question.

These topic tests have been **fully cross-referenced** to the Pearson textbook, *Edexcel AS and A level Further Mathematics Core Pure Mathematics Book 1/AS* (ISBN 978-1292183336), for your convenience (see reference sheet on page 2). Each test has been designed to reflect the specification fully.

About the fundamentals tests

These **fundamentals** tests focus on isolating and testing the core skills of each topic. The questions are designed to use simple numbers and contexts **so that students can show what they can do,** and to allow you to easily identify any weaknesses.

Timings

The recommended times for students to complete each test are given at the top of individual tests.

Calculator use

The effective use of a calculator is one of the objectives of the new specification and is encouraged for all the enclosed tests.

Also available from ZigZag Education

For students who are ready to go beyond the fundamentals, a complete set of **challenge** tests are available. 50% of the marks in these tests come from concepts covered in the fundamentals tests in order to reinforce learning and boost students' confidence, while the other 50% increases in difficulty and progresses the concepts covered.

To prepare students for the exam itself, our **expert** tests contain 25% repeated marks from the fundamentals and challenge tests, and 75% exam-style material with compound/multistep questions.

For each collection of Set A tests we also offer a corresponding collection of Set B duplicated tests with the same styles of questions but different numbers. This allows for a variety of **flexible** uses including:

- **Test → Homework**: Students use test B as a homework to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Homework → Test**: Students revise as homework using test A before doing test B in class under test conditions.
- **Test → Classwork**: Students work through test B with teacher input to consolidate on areas of weakness identified from completing test A under test conditions in class.
- **Classwork → Test**: Students work through test A with teacher input, before checking their learning by completing test B under test conditions.

For total flexibility, the Set A and Set B tests of all three levels can be run on a rolling basis, using the fundamentals tests as starters, with a time interval between them, leaving one expert level test to use at the end of the course for topic revision.

Cross-referencing Grid

Topic	Edexcel spec. points	Subtopics	Chapter Reference - Edexcel Pearson textbook [ISBN: 9781292183336]
Complex Numbers	2.1–2.7	Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram	1, 2
Algebra and Functions	4.1–4.3	Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots	3, 4
Volumes of Revolution	5.1	Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes	5
Matrices	3.1–3.8	Matrices, matrix multiplication, determinants, inverting 2×2 and 3×3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation	6, 7
Proof by Induction	1.1	Proof by mathematical induction, proving divisibility results, proving statements involving matrices	8
Further Vectors	6.1–6.5	Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars	9

Complex Numbers – Test A (45 mins) ✓**Fundamentals** Challenge Expert

Subtopics: Imaginary and complex numbers, multiplying complex numbers, complex conjugation, roots of quadratic equations, solving cubic and quartic equations, Argand diagrams, modulus and argument, modulus-argument form of complex numbers, loci and regions in the Argand diagram

1. Write each of the following in the form bi, where b is a real number given in **simplified surd form** where appropriate:
   a) $\sqrt{-25}$ b) $\sqrt{-16}$ c) $\sqrt{-18}$ **[3]**

2. Given that $z_1 = 3+2\text{i}$ and $z_2 = 4-\text{i}$, write each of the following in the form $a + b\text{i}$, where $a,b \in \mathbb{R}$:
   a) $z_1 + z_2$ b) $5z_1 - z_2$ c) $z_1 z_2$ **[5]**

3. Given that $z = a+b\text{i}$ and $w = a-b\text{i}$, where $a,b \in \mathbb{R}$, explain why $z+w$ is **always real**. **[1]**

4. Solve each of the following equations by **completing the square**. Give your answers in the form $a \pm b\text{i}$, where $a,b \in \mathbb{R}$.
   a) $z^2 - 8z + 52 = 0$ b) $z^2 + 4z + 85 = 0$ **[6]**

5. Find z^*, $z + z^*$ and zz^* for each of the following:
   a) $z = 5+2\text{i}$ b) $z = 2-4\text{i}$ **[8]**

6. **Simplify** $(1-2\text{i})^3$. Give your answer in the form $a + b\text{i}$, where $a,b \in \mathbb{R}$. **[4]**

7. Write each of the following in the form $a + b\text{i}$, where a and b are rational numbers in their lowest terms:
   a) $\dfrac{1+2\text{i}}{3-\text{i}}$ b) $\dfrac{5-\text{i}}{2+2\text{i}}$ c) $\dfrac{4+2\text{i}}{3-2\text{i}}$ **[6]**

8. $f(z) = z^3 - 8z^2 + 29z - 52$
   a) **Verify that** $f(4) = 0$ **[1]**
   b) **Hence** solve $f(z) = 0$ **completely** **[4]**

9. $z_1 = 4+8\text{i}$ and $z_2 = 6-4\text{i}$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram. **[4]**

10. Express each of the following in the form $r(\cos\theta + \text{i}\sin\theta)$, where θ is in radians and $-\pi < \theta \le \pi$.
    Where appropriate, give θ either in terms of π or to **3 significant figures**, and r in **simplified surd form**.
    a) $3+4\text{i}$ b) $6-2\text{i}\sqrt{3}$ c) $-2+6\text{i}$ **[9]**

11. On separate Argand diagrams, **sketch the locus** of z for each of the following:
    a) $|z-2| = 2$ b) $|z-3+\text{i}| = 3$ c) $|z-3\text{i}| = |z-3|$ **[6]**

12. **Shade the region** on an Argand diagram represented by $2 \le |z-4+2\text{i}| \le 5$ **[3]**

TOTAL 60 MARKS

Subtopics: Sums of natural numbers, sums of squares and cubes, roots of polynomials, linear transformations of roots

1. Evaluate the following sums, showing your reasoning in each case:
   a) $\sum_{r=1}^{5}(r-1)$ **[2]**
   b) $\sum_{r=1}^{48} r$ **[2]**
   c) $\sum_{r=1}^{70}(3r-2)$ **[2]**
   d) $\sum_{r=75}^{100} r$ **[3]**

2. Find an expression for $\sum_{r=1}^{2n} r$ in terms of n, simplifying your answer as far as possible. **[2]**

3. Given that $\sum_{r=1}^{n}(ar+2)=\frac{1}{2}n(3n+7)$ for all $n \geq 1$, find the value of the constant a **[4]**

4. Show that $\sum_{r=1}^{3n} r^2=\frac{1}{2}n(An+1)(Bn+1)$ for some integers A and B, where $A < B$

   You may use the formula $\sum_{r=1}^{n} r^2=\frac{1}{6}n(n+1)(2n+1)$ **[2]**

5. Evaluate $\sum_{r=8}^{24} r^3$ using the formula $\sum_{r=1}^{n} r^3=\frac{1}{4}n^2(n+1)^2$ **[2]**

6. The quadratic equation $ax^2 + bx + c = 0$ has roots $\alpha = 1$ and $\beta = -3$
   Find **integer** values for a, b and c **[3]**

7. The cubic equation $2x^3 - 6x^2 + 2x - 1 = 0$ has roots α, β and γ. **Without solving the equation**, find:
   a) $\alpha+\beta+\gamma$ **[2]**
   b) $\alpha\beta+\alpha\gamma+\beta\gamma$ **[2]**
   c) $\alpha\beta\gamma$ **[2]**

8. The quadratic equation $(k-1)x^2 + (k-3)x + 7 = 0$ has roots α and β. Given that $\alpha + \beta = 1$, find k. **[3]**

9. The quartic equation $4x^4 - 12x^3 + 7x^2 + 5x + 2 = 0$ has roots α, β, γ and δ. **Without solving the equation**, find:
   a) $\alpha+\beta+\gamma+\delta$ **[2]**
   b) $\alpha\beta\gamma\delta$ **[2]**

10. The cubic equation $x^3 + x^2 - 2 = 0$ has roots α, β and γ. These roots satisfy $\alpha + \beta + \gamma = -1$, $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ and $\alpha\beta\gamma = 2$. **Without solving the equation**:
    a) find a cubic equation that has roots 2α, 2β and 2γ **[3]**
    b) find a cubic equation that has roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$ **[3]**

TOTAL 41 MARKS

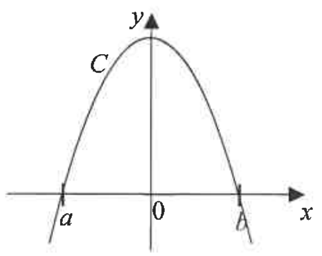
Subtopics: Volumes of revolution with Cartesian equations, adding and subtracting volumes, modelling with volumes

1. Use **calculus** to find the volume of the solid generated when each of the following curves is rotated through 360° about the **x-axis** between the given limits. Give your answers in terms of π.
   a) $y = 5x^2$ between $x = 0$ and $x = 1$ **[3]**
   b) $y = \frac{1}{x}$ between $x = 1$ and $x = 4$ **[3]**
   c) $y = \sqrt{1+x^2}$ between $x = 0$ and $x = 3$ **[3]**

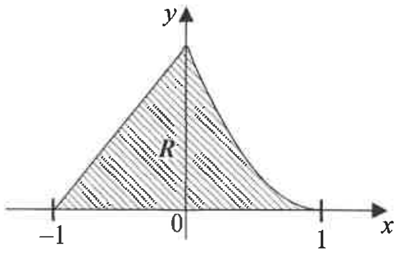
2. Use **calculus** to find the volume of the solid generated when each of the following curves is rotated through 2π radians about the **y-axis** between the given limits. Give your answers to **3 significant figures**.
   a) $x = 3y^4$ between $y = 0$ and $y = 1$ **[3]**
   b) $x = \sqrt[3]{y}$ between $y = 0$ and $y = 8$ **[3]**
   c) $x = y + \frac{1}{y}$ between $y = 1$ and $y = \pi$ **[3]**

3. The curve $y = \lambda x^2$, where λ is a positive constant, is rotated through 360° about the **x-axis** between $x = -2$ and $x = 3$. This generates a solid of revolution with volume 220π. Find the value of λ. **[5]**

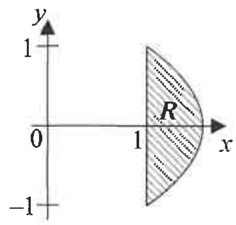
4. The curve C has equation $y = 1 - x^2$. C intersects the x-axis at the points $(a, 0)$ and $(b, 0)$ with $a < b$, as illustrated to the right.
   a) Find the value of a and the value of b **[1]**
   b) Find the volume of the solid generated when C is rotated through 2π radians about the **x-axis** between $x = a$ and $x = b$ in the form $k\pi$, where k is a **rational number** **[3]**



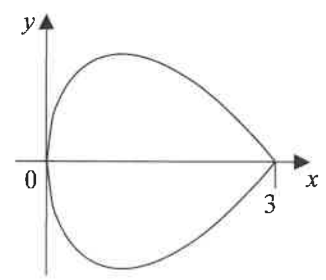
5. The shaded region R shown to the right is bounded by the x-axis, the line $y = 1 + x$ for $-1 \le x \le 0$, and the curve $y = (x-1)^2$ for $0 \le x \le 1$.
Find the volume of the solid generated when R is rotated through 360° about the **x-axis**. Give your answer in the form $k\pi$ for a **rational** number k. **[6]**



6. The finite region R is bounded by the line $x = 1$ and the curve with equation $x = 2 - y^2$ for $-1 \le y \le 1$, as shown to the right. Find the volume of the solid generated when R is rotated through 2π radians about the **y-axis**. Give your answer to **3 significant figures**. **[7]**



7. The cross section of a baby turnip is modelled by the equation $y^2 = x^3 - 6x^2 + 9x$ for $0 \le x \le 3$, as sketched to the right, where each unit on the axes represents 1 cm. Use this model to estimate the volume of the baby turnip to **2 significant figures**. **[3]**



TOTAL 43 MARKS

Subtopics: Matrices, matrix multiplication, determinants, inverting 2 × 2 and 3 × 3 matrices, solving systems of equations using matrices, linear transformations in two and three dimensions, reflections and rotations, enlargements and stretches, successive transformations, the inverse of a linear transformation

1. For the matrices $\mathbf{A}=\begin{pmatrix}2 & 3\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & 1\\ 2 & 1\end{pmatrix}$, **without** using a calculator, find:

   a) $\mathbf{A}+\mathbf{B}$ b) $\mathbf{B}-\mathbf{A}$ c) $3\mathbf{A}$ **[3]**

2. Given that $\mathbf{A}=\begin{pmatrix}3 & -2\\ -1 & 0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$, **without** using a calculator, find:

   a) $\mathbf{AB}$ b) $\mathbf{BA}$ c) $\mathbf{A}^2$ **[6]**

3. For the matrix $\mathbf{M}=\begin{pmatrix}3 & -5\\ -2 & 4\end{pmatrix}$, **without** using a calculator:

   a) find $\mathbf{M}^{\mathrm{T}}$ **[1]**

   b) find $\mathbf{M}^{-1}$ and verify that $\mathbf{M}^{-1}\mathbf{M}=\mathbf{I}$, where **I** is the 2 × 2 identity matrix **[5]**

4. For each of the matrices $\mathbf{A}=\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & -2 & 1\end{pmatrix}$, **without** using a calculator:

   a) find the determinant **[7]**

   b) find the inverse if the matrix is **non-singular**, showing your working **[7]**

5. Three planes *A*, *B* and *C* are defined by the equations:

   $A:\ 2x+3y+3z=10$

   $B:\ 3x+y-z=2$

   $C:\ -x+4y+4z=6$

   By constructing and solving a suitable matrix equation, show that the planes intersect at a **single point** *P*, and find the coordinates of *P* **[5]**

6. Identify which of the following are linear transformations. For those that **are**, give their matrix representations. For those that are **not**, explain why not.

   a) $P:\ \begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}x+3y\\ x-y\end{pmatrix}$ b) $Q:\ \begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}2x\\ y+1\end{pmatrix}$ c) $R:\ \begin{pmatrix}x\\ y\end{pmatrix}\mapsto\begin{pmatrix}0\\ x+3y\end{pmatrix}$ **[6]**

7. *R* is a reflection in the line $x=0$

   S is a stretch with scale factor –3 parallel to the *x*-axis and scale factor 3 parallel to the *y*-axis.

   A triangle has vertices at $A=(0, 4)$, $B=(-3, 4)$ and $C=(1, 1)$

   a) Find matrix representations of *R* and *S* **[2]**

   b) Find the images of the vertices of this triangle under the transformation *R* **[3]**

   c) Identify the **single geometric transformation** represented by *R* followed by *S* **[3]**

8. Find the **image** of the point (–2, 4, 0) under a rotation by angle 60° anticlockwise about the *x*-axis. **[4]**

9. The matrix $\mathbf{M}=\begin{pmatrix}4 & 0\\ 0 & -4\end{pmatrix}$ represents a linear transformation *T*. Given that *T* maps point *P* with coordinates (*x*, *y*) onto point P' with coordinates (12, –4), find the coordinates of *P*. **[4]**

TOTAL 56 MARKS

Subtopics: Proof by mathematical induction, proving divisibility results, proving statements involving matrices

1. Consider the statement: $\sum_{r=1}^{n}(2r-1)=n^2$

   a) Show that this statement is true when $n = 1$ **[2]**

   b) Given that $\sum_{r=1}^{99}(2r-1)=99^2$, show that $\sum_{r=1}^{100}(2r-1)=100^2$ **[2]**

2. Prove by induction that $\sum_{r=1}^{n}(r-1)=\frac{1}{2}n(n-1)$ for all positive integers n **[5]**

3. Prove by induction that $\sum_{r=1}^{n}(3r^2+3r)=n(n+1)(n+2)$ for all positive integers n **[6]**

4. Bob tries to prove that $\sum_{r=1}^{n}(r-1)=\frac{1}{2}\left(n-\frac{1}{2}\right)^2$ for all positive integers n. His attempt is shown below.

> Assume the statement is true when $n = k$, so $\sum_{r=1}^{k}(r-1)=\frac{1}{2}\left(k-\frac{1}{2}\right)^2$
>
> When $n = k + 1$, the sum becomes:
>
> $$\sum_{r=1}^{k+1}(r-1)=\sum_{r=1}^{k}(r-1)+((k+1)-1)$$
> $$=\frac{1}{2}\left(k-\frac{1}{2}\right)^2+k$$
> $$=\frac{1}{2}k^2-\frac{1}{2}k+\frac{1}{8}+k$$
> $$=\frac{1}{2}k^2+\frac{1}{2}k+\frac{1}{8}$$
> $$=\frac{1}{2}\left(k^2+k+\frac{1}{4}\right)$$
> $$=\frac{1}{2}\left(k+\frac{1}{2}\right)^2$$
> $$=\frac{1}{2}\left((k+1)-\frac{1}{2}\right)^2$$
>
> This is the formula we want to prove but with $n = k + 1$. Hence, by the principle of mathematical induction, the statement is true for all positive integers n.

   Identify the error that Bob has made. **[1]**

5. Let $f(n) = 4^n + 5$. Prove by induction that, for all positive integers n, $f(n)$ is divisible by 3. **[6]**

6. Let $\mathbf{M}=\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$. Prove by induction that $\mathbf{M}^n=\begin{pmatrix}1 & 0\\ 2n & 1\end{pmatrix}$ for every positive integer n. **[6]**

TOTAL 28 MARKS

Subtopics: Equation of a line in three dimensions, equation of a plane in three dimensions, scalar product, calculating angles between lines and planes, points of intersection, finding perpendiculars

1. a) Write down a **vector equation** of the straight line that:
   i) **passes through** the point with position vector $4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$ and is **parallel** to the vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ **[1]**
   ii) **passes through** the point with position vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and is **parallel** to the vector $\begin{pmatrix} -7 \\ -1 \\ 4 \end{pmatrix}$ **[1]**

   b) Write down a **Cartesian equation** in the form $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$ for each of these lines **[2]**

2. Find, in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, an equation of the **plane** that passes through the points $A(4, 2, 1)$, $B(5, 4, 4)$ and $C(3, 2, 0)$ **[3]**

3. The plane Π has Cartesian equation $4x + 3y - 5z = 0$. Show that Π passes through the following points:
   a) (1, 2, 2) b) (4, 3, 5) **[2]**

4. Find the scalar product **a.b** when:
   a) $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$ **[2]**
   b) $\mathbf{a} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$ **[2]**

5. Let $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$
   a) Show that $|\mathbf{a}| = 7$ and $|\mathbf{b}| = 9$ **[3]**
   b) Find the acute angle in degrees between **a** and **b** to **3 significant figures** **[4]**

6. The lines with vector equations $\mathbf{r} = (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \lambda(4\mathbf{i} - \mathbf{j} - \mathbf{k})$ and $\mathbf{r} = (5\mathbf{i} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j})$ intersect. Find the **acute** angle between these lines correct to **the nearest degree**. **[3]**

7. Find, in the form $\mathbf{r}.\mathbf{n} = k$, an equation of the plane Π that passes through the point (5, –2, 3) and that is perpendicular to the vector $\mathbf{n} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$ **[3]**

8. Find the **acute** angle in degrees between the plane with equation $\mathbf{r}.(2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) = 3$ and the plane with equation $\mathbf{r}.(8\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 11$. Give your answer to **3 significant figures**. **[3]**

9. Find the coordinates of the **point of intersection** of the line with equation $\mathbf{r} = (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \lambda(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ with the plane with equation $\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 12$ **[4]**

10. Let l be the line with equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and P be the point (5, 3, 2)

    Find the **exact shortest distance** between l and P **[5]**

TOTAL 38 MARKS

> **Hint:** Use the fact that $i = \sqrt{-1}$, so $i^2 = -1$

1. a) $\sqrt{-25} = \sqrt{25i^2} = 5i$ **A1**
   b) $\sqrt{-16} = \sqrt{16i^2} = 4i$ **A1**
   c) $\sqrt{-18} = \sqrt{(9\times 2)i^2} = \left(3\sqrt{2}\right)i$ **A1** **[3 Marks]**

2. $z_1 = 3+2i,\ z_2 = 4-i$
   a) $z_1 + z_2 = (3+2i)+(4-i) = 7+i$ **A1**
   b) $5z_1 - z_2 = 5(3+2i)-(4-i) = 15+10i-4+i$ **M1**
   $= 11+11i$ **A1**
   c) $z_1 z_2 = (3+2i)(4-i) = 12-3i+8i-2i^2$ **M1**
   $= 12-2(-1)+5i = 14+5i$ **A1** **[5 Marks]**

3. $z = a+bi,\ w = a-bi$
   $z+w = (a+bi)+(a-bi) = 2a+0i$
   The imaginary part of $z+w$ is zero, so $z+w$ is real for any real values of a and b **A1** **[1 Mark]**

4. a) $z^2 - 8z + 52 = 0$
   $(z-4)^2 - 16 + 52 = 0$
   $(z-4)^2 = -36$ **M1**
   $z-4 = \sqrt{-36} = \pm 6i$ **M1**
   $\therefore z = 4 \pm 6i$ **A1**
   b) $z^2 + 4z + 85 = 0$
   $(z+2)^2 - 4 + 85 = 0$
   $(z+2)^2 = -81$ **M1**
   $z+2 = \sqrt{-81} = \pm 9i$ **M1**
   $\therefore z = -2 \pm 9i$ **A1** **[6 Marks]**

> **Hint:** z^* is the complex conjugate of $z = a + bi$, and is equal to $a - bi$

> **Alternatively:** Learn that if $z = a + bi$, then $z + z^* = 2a$, and $zz^* = a^2 + b^2$

5. a) $z = 5+2i$
   $\therefore z^* = 5-2i$ **A1**
   $z + z^* = (5+2i)+(5-2i) = 10$ **A1**
   $zz^* = (5+2i)\times(5-2i) = 25-10i+10i-4i^2$ **M1**
   $= 25+4 = 29$ **A1**
   b) $z = 2-4i$
   $\therefore z^* = 2+4i$ **A1**
   $z + z^* = (2-4i)+(2+4i) = 4$ **A1**
   $zz^* = (2-4i)\times(2+4i) = 4+8i-8i-16i^2$ **M1**
   $= 4+16 = 20$ **A1** **[8 Marks]**

6. $(1-2i)^3 = (1-2i)(1-2i)^2$
   $= (1-2i)(1-2i-2i+4i^2)$ **M1**
   $= (1-2i)(1-4i-4)$
   $= (1-2i)(-3-4i)$ **M1**
   $= -3-4i+6i+8i^2$ **M1**
   $= -3+2i-8$
   $= -11+2i$ **A1** **[4 Marks]**

7. a) $\frac{1+2i}{3-i} = \frac{1+2i}{3-i} \times \frac{3+i}{3+i} = \frac{3+i+6i+2i^2}{9+3i-3i-i^2}$ **M1**

$= \frac{3+7i-2}{9-(-1)} = \frac{1+7i}{10} = \frac{1}{10} + \frac{7}{10}i$ **A1**

> **Technique:** Multiply the numerator and the denominator by the complex conjugate of the denominator to remove the imaginary component from the denominator

b) $\frac{5-i}{2+2i} = \frac{5-i}{2+2i} \times \frac{2-2i}{2-2i} = \frac{10-10i-2i+2i^2}{4-4i+4i-4i^2}$ **M1**

$= \frac{10-12i-2}{4-(-4)} = \frac{8-12i}{8} = 1 - \frac{3}{2}i$ **A1**

c) $\frac{4+2i}{3-2i} = \frac{4+2i}{3-2i} \times \frac{3+2i}{3+2i} = \frac{12+8i+6i+4i^2}{9+6i-6i-4i^2}$ **M1**

$= \frac{12+14i-4}{9-(-4)} = \frac{8+14i}{13} = \frac{8}{13} + \frac{14}{13}i$ **A1** **[6 Marks]**

8. $f(z) = z^3 - 8z^2 + 29z - 52$

a) **Verify that $f(4) = 0$**

$f(4) = 4^3 - 8\times4^2 + 29\times4 - 52 = 64 - 128 + 116 - 52 = 0$ **A1**

b)

```
           z² − 4z + 13
z − 4 ) z³ − 8z² + 29z − 52
       −(z³ − 4z²)
            −4z² + 29z − 52
          −(−4z² + 16z)
                   13z − 52
                 −(13z − 52)
                          0
```

> **Technique:** $(z - 4)$ is a factor of f(z) by the factor theorem, so f(z) can be written as $(z - 4)(az^2 + bz + c)$. Use long division or inspection to find the quadratic factor.

So $f(z) = (z-4)(z^2 - 4z + 13)$ **M1**

> **Alternatively:** You can solve the quadratic equation by completing the square, as in question 4

Solve $z^2 - 4z + 13 = 0$ by using the quadratic formula with $a = 1$, $b = -4$, $c = 13$:

$z = \frac{4 \pm \sqrt{(-4)^2 - 4\times1\times13}}{2\times1}$ **M1**

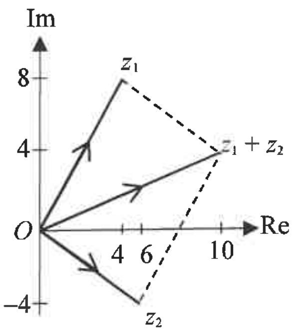
$= \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$ **A1**

> **Tip:** Notice that complex roots always occur as a conjugate pair

So the roots of $f(z)$ are 4, 2 + 3i, 2 − 3i **A1** **[5 Marks]**

9. $z_1 = 4 + 8i$, $z_2 = 6 - 4i$

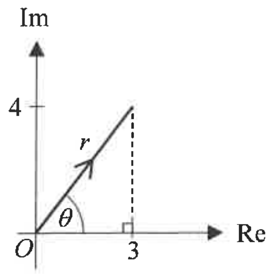
$z_1 + z_2 = (4+8i) + (6-4i) = 10 + 4i$ **A1**



A3 **[4 Marks]**

> **Tip:** $z_1 + z_2$ is represented by the diagonal of the parallelogram with vertices at O, z_1 and z_2

10. a) $z = 3 + 4\text{i}$

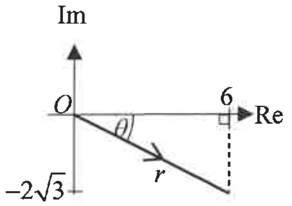


$r = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$ **M1**

$\theta = \arg z = \arctan\left(\frac{4}{3}\right) = 0.927295... = 0.927$ (3 s.f.) **M1**

$\therefore 3 + 4\text{i} = 5(\cos 0.927 + \text{i}\sin 0.927)$ **A1**

b) $z = 6 - 2\text{i}\sqrt{3}$



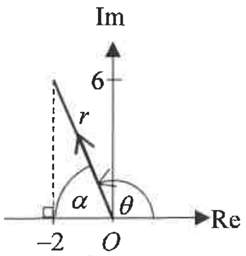
$r = \sqrt{6^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{48} = 4\sqrt{3}$ **M1**

$\theta = \arg z = -\arctan\left(\frac{2\sqrt{3}}{6}\right) = -\frac{\pi}{6}$ **M1**

$\therefore 6 - 2\text{i}\sqrt{3} = 4\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + \text{i}\sin\left(-\frac{\pi}{6}\right)\right)$ **A1**

Tip: Sketch an Argand diagram and consider the lengths in a right-angled triangle, i.e. use positive values in the arctan calculation, and then use your diagram to help you work out the argument so that $-\pi < \theta \leq \pi$

c) $z = -2 + 6\text{i}$



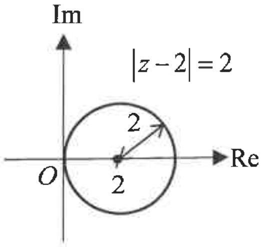
$r = \sqrt{(-2)^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$ **M1**

$\theta = \arg z = \pi - \alpha = \pi - \arctan\left(\frac{6}{2}\right)$

$= \pi - 1.24904... = 1.89254... = 1.89$ (3 s.f.) **M1**

$\therefore -2 + 6\text{i} = 2\sqrt{10}(\cos 1.89 + \text{i}\sin 1.89)$ **A1** **[9 Marks]**

11. a)

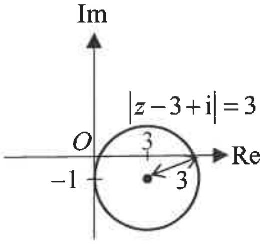


A1 circle with centre (2, 0)
A1 radius 2

Hint: The locus of $|z-a-bi|=r$ is a circle of radius r and centre (a, b)

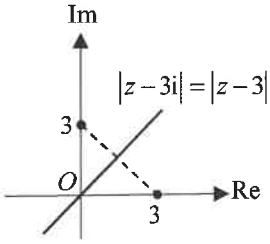
Alternatively: Replace z with $x + iy$, group the real parts and group the imaginary parts, then calculate the modulus to form the equation of a circle

b)



A1 circle with centre (3, −1)
A1 radius 3

c)

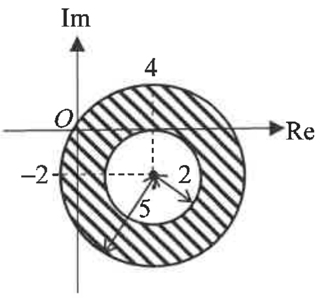


A1 perpendicular bisector
A1 (0, 3) and (3, 0) shown

Technique: The locus of $|z-z_1| = |z-z_2|$ is the perpendicular bisector of the line joining the points representing z_1 and z_2

[6 Marks]

12.



A1 circle with centre (4, −2) and radius 2
A1 circle with centre (4, −2) and radius 5
A1 area between two circles shaded

[3 Marks]

TOTAL 60 MARKS

1\. a) $\sum_{r=1}^{5}(r-1)=(1-1)+(2-1)+(3-1)+(4-1)+(5-1)$ **M1**

$=0+1+2+3+4$

$=10$ **A1**

> **Tip:** When a sum only contains a few terms, you can add them by hand rather than using the formulae

b) $\sum_{r=1}^{48}r=\frac{1}{2}\times48\times(48+1)$ **M1**

$=1176$ **A1**

> **Technique:** The sum of the first n natural numbers is given by the formula $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$

c) $\sum_{r=1}^{70}(3r-2)=3\sum_{r=1}^{70}r-2\sum_{r=1}^{70}1$

$=3\times\frac{1}{2}\times70\times(70+1)-2\times70$ **M1**

$=7315$ **A1**

> **Technique:** You can simplify sums using the rules:
> - $\sum_{r=1}^{n}k\,\mathrm{f}(r)=k\sum_{r=1}^{n}\mathrm{f}(r)$
> - $\sum_{r=1}^{n}(\mathrm{f}(r)+\mathrm{g}(r))=\sum_{r=1}^{n}\mathrm{f}(r)+\sum_{r=1}^{n}\mathrm{g}(r)$

d) $\sum_{r=75}^{100}r=\sum_{r=1}^{100}r-\sum_{r=1}^{74}r$ **M1**

$=\frac{1}{2}\times100\times(100+1)-\frac{1}{2}\times74\times(74+1)$ **M1**

$=2275$ **A1** **[9 Marks]**

> **Tip:** If a sum starts at $m>1$, then you can use the formula $\sum_{r=m}^{n}\mathrm{f}(r)=\sum_{r=1}^{n}\mathrm{f}(r)-\sum_{r=1}^{m-1}\mathrm{f}(r)$

2\. $\sum_{r=1}^{2n}r=\frac{1}{2}\times2n\times(2n+1)$ **M1**

$=n(2n+1)$ **A1** **[2 Marks]**

> **Technique:** Use the formula $\sum_{r=1}^{n}r=\frac{1}{2}n(n+1)$, but replace n with $2n$

3\. $\sum_{r=1}^{n}(ar+2)=a\sum_{r=1}^{n}r+2\sum_{r=1}^{n}1$

$=a\times\frac{1}{2}\times n(n+1)+2\times n$ **M1**

$=\frac{a}{2}n(n+1)+2n$

$=\frac{1}{2}n(a(n+1)+4)$

$=\frac{1}{2}n(an+a+4)$ **A1**

> **Technique:** Write the sum as a polynomial in n. Then you can compare the coefficients of this polynomial to the one given in the question to determine a.

We are told in the question that $\sum_{r=1}^{n}(ar+2)=\frac{1}{2}n(3n+7)$, so $\frac{1}{2}n(an+a+4)=\frac{1}{2}n(3n+7)$ **M1**

We can see by comparing coefficients of n within each bracket that $a=3$ **A1** **[4 Marks]**

4\. **Show that $\sum_{r=1}^{3n}r^2=\frac{1}{2}n(An+1)(Bn+1)$ for some integers A and B where $A<B$**

$\sum_{r=1}^{3n}r^2=\frac{1}{6}\times3n(3n+1)(2\times3n+1)$ **M1**

$=\frac{1}{2}n(3n+1)(6n+1)$ **A1**

(So $A=3$ and $B=6$) **[2 Marks]**

> **Technique:** Replace n with $3n$ in the formula: $\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$

5\. $\sum_{r=8}^{24}r^3=\sum_{r=1}^{24}r^3-\sum_{r=1}^{7}r^3$

$=\frac{1}{4}\times24^2\times(24+1)^2-\frac{1}{4}\times7^2\times(7+1)^2$ **M1** **[2 Marks]**

$=90000-784=89216$ **A1**

> **Technique:** The sum of the first n cubes is: $\sum_{r=1}^{n}r^3=\frac{1}{4}n^2(n+1)^2$

6. $ax^2 + bx + c = 0$ has roots $\alpha = 1$ and $\beta = -3$

So $-\frac{b}{a} = \alpha + \beta = 1 - 3 = -2$ **M1**

Also $\frac{c}{a} = \alpha\beta = 1 \times (-3) = -3$ **M1**

Rearranging these gives the equations $b = 2a$ and $c = -3a$
So one integer solution is $a = 1$, $b = 2$, $c = -3$ **A1**
[Also allow any other integer solutions with $b = 2a$ and $c = -3a$] **[3 Marks]**

Technique: If $ax^2 + bx + c = 0$ has roots α and β, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Alternatively: Multiply out the expression $(x - 1)(x + 3)$ to get the required polynomial

7. $2x^3 - 6x^2 + 2x - 1 = 0$ has roots α, β and γ

a) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= -\frac{-6}{2}$ **M1**

$= 3$ **A1**

b) $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= \frac{2}{2}$ **M1**

$= 1$ **A1**

c) $\alpha\beta\gamma = -\frac{d}{a}$

$= -\frac{-1}{2}$ **M1**

$= \frac{1}{2}$ **A1** **[6 Marks]**

Technique: If the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ, then:

- $\alpha + \beta + \gamma = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

8. $(k - 1)x^2 + (k - 3)x + 7 = 0$ has roots α and β

$\alpha + \beta = -\frac{b}{a} = -\frac{(k-3)}{k-1}$ **M1**

We are told in the question that $\alpha + \beta = 1$, so $-\frac{(k-3)}{k-1} = 1$ **M1**

$\therefore -(k-3) = k - 1$
$\therefore 2k = 4$
So $k = 2$ **A1** **[3 Marks]**

9. $4x^4 - 12x^3 + 7x^2 + 5x + 2 = 0$ has roots α, β, γ and δ

a) $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

$= -\frac{-12}{4}$ **M1**

$= 3$ **A1**

b) $\alpha\beta\gamma\delta = \frac{e}{a}$

$= \frac{2}{4}$ **M1**

$= \frac{1}{2}$ **A1** **[4 Marks]**

Technique: If the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ has roots α, β, γ and δ, then:

- $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

10. $x^3 + x^2 - 2 = 0$ has roots α, β and γ where $\alpha + \beta + \gamma = -1$, $\alpha\beta + \alpha\gamma + \beta\gamma = 0$ and $\alpha\beta\gamma = 2$

a) Let the equation with roots 2α, 2β and 2γ be $w^3 + pw^2 + qw + r = 0$

Then, $-p = 2\alpha + 2\beta + 2\gamma = 2(\alpha + \beta + \gamma) = 2 \times (-1) = -2$, so $p = 2$ **M1**

$q = (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma) = 4(\alpha\beta + \alpha\gamma + \beta\gamma) = 4 \times 0 = 0$,

So $q = 0$

$-r = (2\alpha)(2\beta)(2\gamma) = 8(\alpha\beta\gamma) = 8 \times 2 = 16$, so $r = -16$ **A1**

And so the equation is $w^3 + 2w^2 - 16 = 0$ **A1**

[Allow any multiple of the left-hand side in this equation]

Tip: Use a different letter for the variable in your new equation to avoid confusion. Here we use w.

Alternatively: Let $p(x) = x^3 + x^2 - 2$, which has roots $x = \alpha, \beta$ and γ. Then $p\left(\frac{w}{2}\right)$ has roots $w = 2\alpha, 2\beta$ and 2γ. Expand $p\left(\frac{w}{2}\right)$ to obtain a new polynomial in terms of w with the required roots.

b) Let the equation with roots $\alpha + 1$, $\beta + 1$ and $\gamma + 1$ be $y^3 + sy^2 + ty + u = 0$

Then, $-s = (\alpha + 1) + (\beta + 1) + (\gamma + 1) = (\alpha + \beta + \gamma) + 3 = -1 + 3 = 2$, so $s = -2$ **M1**

$$\begin{aligned} t &= (\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\beta+1)(\gamma+1) \\ &= \alpha\beta + \alpha + \beta + 1 + \alpha\gamma + \alpha + \gamma + 1 + \beta\gamma + \beta + \gamma + 1 \\ &= (\alpha\beta + \alpha\gamma + \beta\gamma) + 2(\alpha + \beta + \gamma) + 3 \\ &= 0 + 2 \times (-1) + 3 \\ &= 1 \end{aligned}$$

So $t = 1$

Alternatively: Let $p(x) = x^3 + x^2 - 2$, which has roots $x = \alpha, \beta$ and γ. Then $p(w - 1)$ has roots $w = \alpha + 1, \beta + 1$ and $\gamma + 1$. Expand $p(w - 1)$ to obtain a new polynomial in terms of w with the required roots.

$$\begin{aligned} -u &= (\alpha+1)(\beta+1)(\gamma+1) \\ &= (\alpha+1)(\beta\gamma + \beta + \gamma + 1) \\ &= \alpha\beta\gamma + \alpha\beta + \alpha\gamma + \alpha + \beta\gamma + \beta + \gamma + 1 \\ &= \alpha\beta\gamma + (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) + 1 \\ &= 2 + 0 + (-1) + 1 \\ &= 2 \end{aligned}$$

so $u = -2$ **A1**

And so the equation is $y^3 - 2y^2 + y - 2 = 0$ **A1**

[Allow any multiple of the left-hand side in this equation] **[6 Marks]**

TOTAL 41 MARKS

1. a) The curve has equation $y = 5x^2$

$\therefore V = \pi\int_0^1 (5x^2)^2\,dx$ **M1**

$= \pi\int_0^1 25x^4\,dx$

$= \pi\left[5x^5\right]_0^1$ **A1**

$= \pi(5\times1 - 5\times0)$

$= 5\pi$ **A1**

> **Technique:** If the curve with equation $y = f(x)$ is rotated 360° about the x-axis between $x = a$ and $x = b$, then the volume of the solid generated is given by $V = \pi\int_a^b y^2\,dx = \pi\int_a^b (f(x))^2\,dx$

b) The curve has equation $y = \dfrac{1}{x}$

$\therefore V = \pi\int_1^4 \left(\dfrac{1}{x}\right)^2 dx$ **M1**

$= \pi\int_1^4 x^{-2}\,dx$

$= \pi\left[-\dfrac{1}{x}\right]_1^4$ **A1**

$= \pi\left(-\dfrac{1}{4} + \dfrac{1}{1}\right)$

$= \dfrac{3\pi}{4}$ **A1**

> **Tip:** The question tells you to use calculus, so you cannot simply use your calculator to evaluate the integral. However, once you have an answer you *can* use your calculator to check that it is correct.

c) The curve has equation $y = \sqrt{1+x^2}$

$\therefore V = \pi\int_0^3 \left(\sqrt{1+x^2}\right)^2 dx$ **M1**

$= \pi\int_0^3 (1+x^2)\,dx$

$= \pi\left[x + \dfrac{1}{3}x^3\right]_0^3$ **A1**

$= \pi\left(3 + \dfrac{1}{3}\times27 - 0 - \dfrac{1}{3}\times0\right)$

$= 12\pi$ **A1**

[9 Marks]

2. a) The curve has equation $x = 3y^4$

$\therefore V = \pi\int_0^1 (3y^4)^2\,dy$ **M1**

$= \pi\int_0^1 9y^8\,dy$

$= \pi\left[y^9\right]_0^1$ **A1**

$= \pi(1-0)$

$= 3.14159\ldots = 3.14$ (3 s.f.) **A1**

> **Technique:** If the curve with equation $x = f(y)$ is rotated 2π radians about the y-axis between $y = a$ and $y = b$, then the volume of the solid generated is given by $V = \pi\int_a^b x^2\,dy = \pi\int_a^b (f(y))^2\,dy$

b) The curve has equation $x = \sqrt[3]{y}$

$\therefore V = \pi\int_0^8 \left(\sqrt[3]{y}\right)^2 dy$ **M1**

$= \pi\int_0^8 y^{\frac{2}{3}}\,dy$

> **Tip:** Remember that $\sqrt[3]{y} = y^{1/3}$

$= \pi\left[\dfrac{3}{5}y^{\frac{5}{3}}\right]_0^8$ **A1**

$= \pi\left(\dfrac{3}{5}\times32 - \dfrac{3}{5}\times0\right)$

$= 60.3185\ldots = 60.3$ (3 s.f.) **A1**

c) The curve has equation $x = y + \frac{1}{y}$

$$\therefore V = \pi \int_1^{\pi} \left(y + \frac{1}{y} \right)^2 \text{d}y \quad \textbf{M1}$$

$$= \pi \int_1^{\pi} \left(y^2 + 2 + \frac{1}{y^2} \right) \text{d}y$$

$$= \pi \left[\frac{1}{3} y^3 + 2y - \frac{1}{y} \right]_1^{\pi} \quad \textbf{A1}$$

$$= \pi \left(\frac{1}{3} \pi^3 + 2\pi - \frac{1}{\pi} - \frac{1}{3} \times 1 - 2 \times 1 + 1 \right)$$

$= 47.0201... = 47.0$ (3 s.f.) **A1** **[9 Marks]**

3. The curve has equation $y = \lambda x^2$

> **Technique:** Find the volume in terms of the unknown λ, then set your expression equal to the given volume and solve the resulting equation in λ

$$\therefore V = \pi \int_{-2}^{3} \left(\lambda x^2 \right)^2 \text{d}x \quad \textbf{M1}$$

$$= \pi \int_{-2}^{3} \lambda^2 x^4 \, \text{d}x$$

$$= \pi \left[\frac{1}{5} \lambda^2 x^5 \right]_{-2}^{3} \quad \textbf{A1}$$

$$= \pi \left(\frac{1}{5} \lambda^2 \times 243 - \frac{1}{5} \lambda^2 \times (-32) \right)$$

$$= 55\pi\lambda^2 \quad \textbf{A1}$$

We are told in the question that $V = 220\pi$, so $55\pi\lambda^2 = 220\pi$ **M1**

Hence $\lambda^2 = \frac{220\pi}{55\pi} = 4$ and so $\lambda = \pm 2$

We are told in the question that $\lambda > 0$, so $\lambda = 2$ **A1** **[5 Marks]**

4. a) C has equation $y = 1 - x^2$, so C intersects the x-axis when $1 - x^2 = 0$

$\therefore \; x = \pm\sqrt{1}$

So $a = -1$ and $b = 1$ **A1**

b) $V = \pi \int_{-1}^{1} \left(1 - x^2 \right)^2 \text{d}x$ **M1**

$$= \pi \int_{-1}^{1} \left(1 - 2x^2 + x^4 \right) \text{d}x$$

$$= \pi \left[x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right]_{-1}^{1} \quad \textbf{A1}$$

$$= \pi \left(1 - \frac{2}{3} \times 1 + \frac{1}{5} \times 1 - (-1) + \frac{2}{3} \times (-1) - \frac{1}{5} \times (-1) \right)$$

$$= \frac{16}{15} \pi \quad \textbf{A1}$$

$\left(\text{So } k = \frac{16}{15} \right)$ **[4 Marks]**

5. Split R into two regions, R_1 for $-1 \le x \le 0$ and R_2 for $0 < x \le 1$. Let V_1 be the volume of the solid generated when R_1 is rotated 360° about the x-axis, and V_2 be the volume of the solid generated when R_2 is rotated 360° about the x-axis.

$V_1 = \pi\int_{-1}^{0}(1+x)^2\,dx$ **M1**

$= \pi\int_{-1}^{0}(1+2x+x^2)\,dx$

$= \pi\left[x+x^2+\frac{1}{3}x^3\right]_{-1}^{0}$ **A1**

$= \pi\left(0+0+\frac{1}{3}\times 0-(-1)-1-\frac{1}{3}\times(-1)\right)$

$= \frac{1}{3}\pi$ **A1**

Alternatively: The solid generated by rotating R_1 360° about the x-axis is a cone of height $h = 1$ and radius $r = 1$. Using the formula for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, gives the answer.

$V_2 = \pi\int_{0}^{1}\left((x-1)^2\right)^2\,dx$

$= \pi\int_{0}^{1}(x-1)^4\,dx$

$= \pi\int_{0}^{1}(x^4-4x^3+6x^2-4x+1)\,dx$

$= \pi\left[\frac{1}{5}x^5-x^4+2x^3-2x^2+x\right]_{0}^{1}$ **A1**

$= \pi\left(\frac{1}{5}\times 1-1+2\times 1-2\times 1+1-\frac{1}{5}\times 0+0-2\times 0+2\times 0-0\right)$

$= \frac{1}{5}\pi$ **A1**

The total volume, V, of the solid generated when the whole region R is rotated 360° about the x-axis is the sum of these two volumes:

$V = V_1 + V_2 = \frac{1}{3}\pi + \frac{1}{5}\pi = \frac{8}{15}\pi$ **A1** **[6 Marks]**

6. Let V_1 be the volume of the solid generated by rotating the area between the curve $x = 2 - y^2$ and the y-axis between $y = -1$ and $y = 1$ through 2π radians about the y-axis.
Let V_2 be the volume of the solid generated by rotating the area between the line $x = 1$ and the y-axis between $y = -1$ and $y = 1$ through 2π radians about the y-axis.
The volume, V, of the solid generated by rotating R through 2π radians about the y-axis is given by $V = V_1 - V_2$

Hint: Some volumes are easier to compute by first calculating the volume of a larger solid, and then subtracting the part that is not needed

$V_1 = \pi\int_{-1}^{1}(2-y^2)^2\,dy$ **M1**

$= \pi\int_{-1}^{1}(4-4y^2+y^4)\,dy$

$= \pi\left[4y-\frac{4}{3}y^3+\frac{1}{5}y^5\right]_{-1}^{1}$ **A1**

$= \pi\left(4\times 1-\frac{4}{3}\times 1+\frac{1}{5}\times 1-4\times(-1)+\frac{4}{3}\times(-1)-\frac{1}{5}\times(-1)\right)$

$= \frac{86}{15}\pi$ **A1**

The solid generated by rotating the area bounded by the lines $x = 1$, $y = -1$ and $y = 1$ through 2π radians about the y-axis is a cylinder of radius $r = 1$ and height $h = 2$. So:

$V_2 = \pi r^2 h$

$= \pi \times 1^2 \times 2$ **M1**

$= 2\pi$ **A1**

and so $V = V_1 - V_2$

$= \frac{86}{15}\pi - 2\pi$ **M1**

$= 11.7286... = 11.7$ (3 s.f.) **A1** **[7 Marks]**

7. The curve has equation $y^2 = x^3 - 6x^2 + 9x$

$V = \pi\int_0^3 y^2 \, dx$

$= \pi\int_0^3 \left(x^3 - 6x^2 + 9x\right) dx$ **M1**

$= \pi\left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2\right]_0^3$ **A1**

$= \pi\left(\frac{1}{4}\times 81 - 2\times 27 + \frac{9}{2}\times 9 - \frac{1}{4}\times 0 + 2\times 0 - \frac{9}{2}\times 0\right)$

$= 21.2057... = 21 \text{ cm}^3$ (2 s.f.) **A1** **[3 Marks]**

Tip: To find the volume of revolution, you need to integrate y^2. Sometimes the equation you are given will already express y^2 in terms of x.

TOTAL 43 MARKS

1. $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$

a) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 2 & 0 \end{pmatrix}$ **A1**

Technique: Add the corresponding elements in each matrix

b) $\mathbf{B} - \mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$ **A1**

Technique: Multiply each element in the matrix by the scalar

c) $3\mathbf{A} = 3\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 0 & -3 \end{pmatrix}$ **A1**

[3 Marks]

2. $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

a) Let $\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a = 3 \times 2 + (-2) \times 0 = 6$

$b = 3 \times 1 + (-2) \times 2 = -1$

$c = (-1) \times 2 + 0 \times 0 = -2$

$d = (-1) \times 1 + 0 \times 2 = -1$ **M1**

So $\mathbf{AB} = \begin{pmatrix} 6 & -1 \\ -2 & -1 \end{pmatrix}$ **A1**

Technique: To find each element in **AB**, find the sum of the elements in each row of **A** multiplied by the corresponding elements in each column of **B**

b) Let $\mathbf{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a = 2 \times 3 + 1 \times (-1) = 5$

$b = 2 \times (-2) + 1 \times 0 = -4$

$c = 0 \times 3 + 2 \times (-1) = -2$

$d = 0 \times (-2) + 2 \times 0 = 0$ **M1**

So $\mathbf{BA} = \begin{pmatrix} 5 & -4 \\ -2 & 0 \end{pmatrix}$ **A1**

c) Let $\mathbf{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Technique: To find $\mathbf{A}^2$, multiply **A** by itself

$a = 3 \times 3 + (-2) \times (-1) = 11$

$b = 3 \times (-2) + (-2) \times 0 = -6$

$c = (-1) \times 3 + 0 \times (-1) = -3$

$d = (-1) \times (-2) + 0 \times 0 = 2$ **M1**

So $\mathbf{A}^2 = \begin{pmatrix} 11 & -6 \\ -3 & 2 \end{pmatrix}$ **A1**

[6 Marks]

3. $\mathbf{M}=\begin{pmatrix}3 & -5\\ -2 & 4\end{pmatrix}$

a) $\mathbf{M}^{\mathrm{T}}=\begin{pmatrix}3 & -2\\ -5 & 4\end{pmatrix}$ **A1**

Hint: $\mathbf{M}^{\mathrm{T}}$ means **M** is transposed, i.e. rows and columns are interchanged

b) **Find $\mathbf{M}^{-1}$ and verify that $\mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$, where I is the 2 × 2 identity matrix**

$\det \mathbf{M}=\begin{vmatrix}3 & -5\\ -2 & 4\end{vmatrix}=3\times 4-(-5)\times(-2)$ **M1**

$= 12 - 10 = 2$ **A1**

Technique: For a 2 × 2 matrix $\mathbf{M}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$,

$\mathbf{M}^{-1}=\frac{1}{\det \mathbf{M}}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$, where

$\det \mathbf{M} = ad - bc\ (\neq 0)$

So $\mathbf{M}^{-1}=\frac{1}{2}\begin{pmatrix}4 & 5\\ 2 & 3\end{pmatrix}\left(\text{or}\begin{pmatrix}2 & \frac{5}{2}\\ 1 & \frac{3}{2}\end{pmatrix}\right)$ **A1**

$\mathbf{M}^{-1}\mathbf{M}=\begin{pmatrix}2 & \frac{5}{2}\\ 1 & \frac{3}{2}\end{pmatrix}\begin{pmatrix}3 & -5\\ -2 & 4\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$a=2\times 3+\frac{5}{2}\times(-2)=1$

$b=2\times(-5)+\frac{5}{2}\times 4=0$

$c=1\times 3+\frac{3}{2}\times(-2)=0$

$d=1\times(-5)+\frac{3}{2}\times 4=1$ **M1**

So $\mathbf{M}^{-1}\mathbf{M}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{I}$ **A1** **[6 Marks]**

4. $\mathbf{A}=\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & -2 & 1\end{pmatrix}$

a) $\det \mathbf{A}=\begin{vmatrix}5 & -2\\ -2 & 1\end{vmatrix}=5\times 1-(-2)\times(-2)$ **M1**

$= 5 - 4 = 1$ **A1**

$\det \mathbf{B}=\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=1\times 4-2\times 2$ **M1**

$= 4 - 4 = 0$ **A1**

$\det \mathbf{C}=\begin{vmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & -2 & 1\end{vmatrix}=1\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}-2\begin{vmatrix}0 & 2\\ -1 & 1\end{vmatrix}+3\begin{vmatrix}0 & 1\\ -1 & -2\end{vmatrix}$ **M1**

$=1(1+4)-2(0+2)+3(0+1)$

$= 1\times 5 - 2\times 2 + 3\times 1$ **M1**

$= 5 - 4 + 3 = 4$ **A1**

Technique: The determinant of a 3 × 3 matrix $\begin{pmatrix}a & b & c\\ d & e & f\\ g & h & i\end{pmatrix}$ is

$a\begin{vmatrix}e & f\\ h & i\end{vmatrix}-b\begin{vmatrix}d & f\\ g & i\end{vmatrix}+c\begin{vmatrix}d & e\\ g & h\end{vmatrix}$

b) **A** is non-singular as det **A** = 1 ≠ 0 **M1**

> **Hint:** A matrix **M** is non-singular if det **M** ≠ 0, and singular if det **M** = 0. A singular matrix (e.g. matrix **B**) has no inverse.

So $\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ **A1**

C is non-singular as det **C** = 4 ≠ 0 **M1**

To find $\mathbf{C}^{-1}$, first find the matrix of minors, **M**:

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 \\ 8 & 4 & 0 \\ 1 & 2 & 1 \end{pmatrix} \quad \mathbf{M1A1}$$

Then find the transposed matrix of cofactors, $\mathbf{D}^{\mathrm{T}}$:

$$\mathbf{D} = \begin{pmatrix} 5 & -2 & 1 \\ -8 & 4 & 0 \\ 1 & -2 & 1 \end{pmatrix} \therefore \mathbf{D}^{\mathrm{T}} = \begin{pmatrix} 5 & -8 & 1 \\ -2 & 4 & -2 \\ 1 & 0 & 1 \end{pmatrix} \quad \mathbf{M1}$$

> **Technique:** Divide the transposed matrix of cofactors by the determinant of the original matrix

$$\text{So } \mathbf{C}^{-1} = \frac{1}{4}\begin{pmatrix} 5 & -8 & 1 \\ -2 & 4 & -2 \\ 1 & 0 & 1 \end{pmatrix} \left(\text{or } \begin{pmatrix} 5/4 & -2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & 0 & 1/4 \end{pmatrix} \right) \quad \mathbf{A1}\ \mathbf{[14\ Marks]}$$

5. $$\begin{aligned} 2x+3y+3z &= 10 \\ 3x+y-z &= 2 \\ -x+4y+4z &= 6 \end{aligned}$$

$$\begin{pmatrix} 2 & 3 & 3 \\ 3 & 1 & -1 \\ -1 & 4 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix} \quad \mathbf{M1}$$

> **Technique:** Write the coefficients as a 3 × 3 matrix, the variables as a 3 × 1 matrix and the constants as another 3 × 1 matrix

If $\mathbf{M} = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 1 & -1 \\ -1 & 4 & 4 \end{pmatrix}$, then the determinant of **M** is 22, so **M** is non-singular

Therefore, there is a unique solution to the system, and the planes intersect at a single point **A1**

To find this point, solve the matrix equation $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix}$, so $\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix}$

Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{22}\begin{pmatrix} 8 & 0 & -6 \\ -11 & 11 & 11 \\ 13 & -11 & -7 \end{pmatrix}\begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix}$ **M1A1**

> **Technique:** Use your calculator to save time in finding the determinant and the inverse of **M**. Then multiply each side of the matrix equation by $\mathbf{M}^{-1}$ to find the values of *x*, *y* and *z*.

$$= \frac{1}{22}\begin{pmatrix} 44 \\ -22 \\ 66 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

So the coordinates of P are (2, −1, 3) **A1** **[5 Marks]**

6. a) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+3y \\ x-y \end{pmatrix}$ is linear **A1**

> **Hint:** Linear transformations involve only terms of the form $ax + by$ for constants a and b

Its matrix representation is $\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ **A1**

b) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y+1 \end{pmatrix}$ is not linear because $y + 1$ cannot be written in the form $ax + by$ for constants a and b **A1A1**

> **Alternatively:** The origin is not mapped to itself, so $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y+1 \end{pmatrix}$ is not linear

c) $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x+3y \end{pmatrix}$ is linear (as $0 \equiv 0x + 0y$) **A1**

Its matrix representation is $\begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}$ **A1** **[6 Marks]**

7. a) R is a reflection in the line $x = 0$

$\mathbf{R} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **A1**

Technique: Consider the effect of each transformation on the points (1, 0) and (0, 1)

S is a stretch with scale factor –3 parallel to the x-axis and scale factor 3 parallel to the y-axis

$\mathbf{S} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$ **A1**

b) $A: \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} (-1)\times 0 + 0\times 4 \\ 0\times 0 + 1\times 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ so $A' = (0,\ 4)$ **A1**

$B: \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} (-1)\times(-3) + 0\times 4 \\ 0\times(-3) + 1\times 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ so $B' = (3,\ 4)$ **A1**

$C: \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (-1)\times 1 + 0\times 1 \\ 0\times 1 + 1\times 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ so $C' = (-1,\ 1)$ **A1**

c) $\mathbf{SR} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **M1**

Hint: Make sure you get these in the correct order: **R** followed by **S** is represented by the matrix **SR** (in this case it is the same as **RS**, although in general they will be different)

$= \begin{pmatrix} (-3)\times(-1) + 0\times 0 & (-3)\times 0 + 0\times 1 \\ 0\times(-1) + 3\times 0 & 0\times 0 + 3\times 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **A1**

This represents an enlargement, centre (0, 0), by scale factor 3 **A1** **[8 Marks]**

8. A rotation by 60° anticlockwise about the x-axis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° \\ 0 & \sin 60° & \cos 60° \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ **A1A1**

Technique: Consider what happens to the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) under this transformation

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 2\sqrt{3} \end{pmatrix}$ so the image of the point is $\left(-2,\ 2,\ 2\sqrt{3}\right)$ **M1A1** **[4 Marks]**

9. $\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$ represents T, which maps $P(x, y)$ to $P'(12, -4)$

$\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$

Alternatively: Multiply $\begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ to find a 2 × 1 matrix in x and y, and set this equal to $\begin{pmatrix} 12 \\ -4 \end{pmatrix}$

The determinant of **M** is –16, and the inverse of **M** is $\mathbf{M}^{-1} = \frac{1}{-16}\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$ **A1**

$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$ so $\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} 12 \\ -4 \end{pmatrix}$

Therefore, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-16}\begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 12 \\ -4 \end{pmatrix}$ **M1**

$= -\frac{1}{16}\begin{pmatrix} -48 \\ -16 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

So the coordinates of P are (3, 1) **A1A1** **[4 Marks]**

TOTAL 56 MARKS

1. $\sum_{r=1}^{n}(2r-1)=n^2$

a) **Show that the statement is true when $n = 1$**

When $n = 1$ the left-hand side is $\sum_{r=1}^{1}(2r-1)=2\times1-1=1$ **A1**

While the right-hand side is $1^2 = 1$ **A1**
And so the statement is true when $n = 1$

b) **Given that $\sum_{r=1}^{99}(2r-1)=99^2$, show that $\sum_{r=1}^{100}(2r-1)=100^2$**

$\sum_{r=1}^{100}(2r-1)=\sum_{r=1}^{99}(2r-1)+(2\times100-1)$ **M1**

$=99^2+199$

$=10\ 000$

$=100^2$ **A1** **[4 Marks]**

2. **Prove by induction that $\sum_{r=1}^{n}(r-1)=\frac{1}{2}n(n-1)$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1}(r-1)=1-1=0$ and $\frac{1}{2}n(n-1)=\frac{1}{2}\times1\times0=0$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k}(r-1)=\frac{1}{2}k(k-1)$ **M1**

When $n = k + 1$:

$\sum_{r=1}^{k+1}(r-1)=\sum_{r=1}^{k}(r-1)+((k+1)-1)$

$=\frac{1}{2}k(k-1)+(k+1)-1$ **M1**

$=\frac{1}{2}k(k-1)+k$

$=\frac{1}{2}k(k-1+2)$

$=\frac{1}{2}(k+1)k=\frac{1}{2}(k+1)((k+1)-1)$ **A1**

So if the statement is true when $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[5 Marks]**

> **Technique:** To prove a statement by induction for all integers $n \geq 1$, first prove the basis step. This is the statement you are trying to prove in the special case that $n = 1$.

> **Technique:** Once the basis step has been proved, carry out the inductive step. Assume the statement is true when $n = k$ and use this to prove that the statement is true when $n = k + 1$.

> **Technique:** Finish off the proof with a conclusion that invokes the principle of mathematical induction

3. **Prove by induction that $\sum_{r=1}^{n}(3r^2+3r)=n(n+1)(n+2)$ for all positive integers n**

When $n = 1$: $\sum_{r=1}^{1}(3r^2+3r)=3\times1^2+3\times1=6$ and $n(n+1)(n+2)=1\times2\times3=6$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\sum_{r=1}^{k}(3r^2+3r)=k(k+1)(k+2)$ **M1**

When $n = k + 1$:

$\sum_{r=1}^{k+1}(3r^2+3r)=\sum_{r=1}^{k}(3r^2+3r)+3(k+1)^2+3(k+1)$

$=k(k+1)(k+2)+3(k+1)^2+3(k+1)$ **M1**

$=(k+1)(k(k+2)+3(k+1)+3)$

$=(k+1)(k^2+5k+6)$ **A1**

$=(k+1)(k+2)(k+3)=(k+1)((k+1)+1)((k+1)+2)$ **A1**

So if the statement is true when $n = k$, then it is true for $n = k + 1$
Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

4. Bob has not carried out the basis step, i.e. he has failed to prove the statement is true when $n = 1$ **B1** **[1 Mark]**

5. **Prove by induction that, for all positive integers n, $f(n) = 4^n + 5$ is divisible by 3**

When $n = 1$: $f(1) = 4^1 + 5 = 9 = 3 \times 3$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $f(k) = 4^k + 5$ is divisible by 3 **M1**

$f(k+1) = 4^{k+1} + 5 = 4 \times 4^k + 5$

So $f(k+1) - f(k) = (4 \times 4^k + 5) - (4^k + 5)$ **M1**

$= 4 \times 4^k - 4^k$

$= 4^k(4-1) = 3 \times 4^k$ **A1**

Hence $f(k+1) = f(k) + 3 \times 4^k$ **A1**

Since $f(k)$ is divisible by 3 and 3×4^k is divisible by 3, their sum is divisible by 3

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for all positive integers n **A1** **[6 Marks]**

Alternatively: For the inductive step you can show that $f(k + 1) = 4f(k) - 15$. Since $f(k)$ and 15 are both divisible by 3, $f(k + 1)$ will be too.

Tip: If a and b are both divisible by λ then $a \pm b$ is divisible by λ

6. **Prove by induction that $\mathbf{M}^n = \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix}$ for every positive integer n**

$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

When $n = 1$: left-hand side $= \mathbf{M}^1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and right-hand side $= \begin{pmatrix} 1 & 0 \\ 2\times 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ **B1**

So the statement is true for $n = 1$

Assume the statement is true for $n = k$, so $\mathbf{M}^k = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}$ **M1**

$$\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M}$$

$$= \begin{pmatrix} 1 & 0 \\ 2k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \quad \textbf{M1}$$

$$= \begin{pmatrix} 1\times1+0\times2 & 1\times0+0\times1 \\ 2k\times1+1\times2 & 2k\times0+1\times1 \end{pmatrix} \quad \textbf{A1}$$

$$= \begin{pmatrix} 1 & 0 \\ 2(k+1) & 1 \end{pmatrix} \quad \textbf{A1}$$

So if the statement is true for $n = k$, then it is true for $n = k + 1$

Since it is true for $n = 1$, by the principle of mathematical induction it is true for every positive integer n **A1** **[6 Marks]**

TOTAL 28 MARKS

Where they occur, λ and μ are scalar parameters

1. a) i) $\mathbf{r} = (4\mathbf{i} - 5\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ **A1**

Also allow equivalent equations, such as $\mathbf{r} = (4 + \lambda)\mathbf{i} + (-5 + 3\lambda)\mathbf{j} + (1 - 2\lambda)\mathbf{k}$ or $\mathbf{r} = \begin{pmatrix}4\\-5\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix}$

> **Technique:** The vector equation of the line that passes through the point with position vector **a** and that is parallel to the vector **b** is **r** = **a** + λ**b**

ii) $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-7\\-1\\4\end{pmatrix}$ **A1**

Also allow equivalent equations, such as $\mathbf{r} = \begin{pmatrix}1-7\lambda\\-\lambda\\1+4\lambda\end{pmatrix}$ or $\mathbf{r} = (\mathbf{i} + \mathbf{k}) + \lambda(-7\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

b) i) $\mathbf{r} = (4\mathbf{i} - 5\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$, so $\frac{x-4}{1} = \frac{y+5}{3} = \frac{z-1}{-2}$ **A1**

ii) $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}-7\\-1\\4\end{pmatrix}$, so $\frac{x-1}{-7} = \frac{y}{-1} = \frac{z-1}{4}$ **A1** **[4 Marks]**

> **Technique:** If a line has vector equation $\mathbf{r} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} + \lambda\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, then it has Cartesian equation $\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$

2. The plane passes through $A(4, 2, 1)$, $B(5, 4, 4)$ and $C(3, 2, 0)$

So A is a point in the plane, and $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are vectors that lie in the plane

> **Technique:** If **a** is the position vector of a point on a plane and **b** and **c** are non-parallel vectors that lie in the plane, then the plane has equation **r** = **a** + λ**b** + μ**c**

$$\overrightarrow{OA} = \begin{pmatrix}4\\2\\1\end{pmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}5\\4\\4\end{pmatrix} - \begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$$ **M1** for finding at least one vector in the plane

So an equation of the plane is $\mathbf{r} = \begin{pmatrix}4\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ **M1A1**

Allow any equation in form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ where **a** is the position vector of a point in the plane, and **b** and **c** are vectors that lie in the plane **[3 Marks]**

3. $\Pi : 4x + 3y - 5z = 0$

a) **Show that Π passes through the point (1, 2, 2)**

If $(x, y, z) = (1, 2, 2)$, then left-hand side $= 4\times1 + 3\times2 - 5\times2 = 0$ = right-hand side, so (1, 2, 2) lies on Π **B1**

b) **Show that Π passes through the point (4, 3, 5)**

If $(x, y, z) = (4, 3, 5)$, then left-hand side $= 4\times4 + 3\times3 - 5\times5 = 0$ = right-hand side, so (4, 3, 5) lies on Π **B1**

[2 Marks]

4. a) $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$, so

$\mathbf{a}.\mathbf{b} = (3\mathbf{i} + \mathbf{j} - 2\mathbf{k}).(7\mathbf{i} - 4\mathbf{j} + 4\mathbf{k})$

$= 3\times7 + 1\times(-4) + (-2)\times4$ **M1**

$= 9$ **A1**

> **Tip:** If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

b) $\mathbf{a} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$, so

$$\mathbf{a.b} = \begin{pmatrix} -5 \\ 0 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ -4 \end{pmatrix}$$
$$= (-5)\times(-3) + 0\times 0 + 4\times(-4) \quad \textbf{M1}$$
$$= -1 \quad \textbf{A1}$$

[4 Marks]

5. $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = 4\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$

a) **Show that $|\mathbf{a}| = 7$ and $|\mathbf{b}| = 9$**

$$|\mathbf{a}| = \sqrt{2^2 + (-3)^2 + 6^2} \quad \textbf{M1}$$
$$= \sqrt{49}$$
$$= 7 \quad \textbf{A1}$$
$$|\mathbf{b}| = \sqrt{4^2 + (-7)^2 + 4^2}$$
$$= \sqrt{81}$$
$$= 9 \quad \textbf{A1}$$

b) Let θ be the angle between **a** and **b**

Then $\mathbf{a.b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$$= 7\times 9\cos\theta \quad \textbf{M1}$$
$$= 63\cos\theta$$

Also, $\mathbf{a.b} = (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}).(4\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$
$$= 2\times 4 + (-3)\times(-7) + 6\times 4 \quad \textbf{M1}$$
$$= 53$$
$$\therefore 63\cos\theta = 53 \quad \textbf{M1}$$
$$\therefore \theta = \arccos\left(\frac{53}{63}\right) = 32.7255...° = 32.7° \text{ (3 s.f.)} \quad \textbf{A1} \quad \textbf{[7 Marks]}$$

Tip: If the angle between the vectors **a** and **b** is θ, then $\mathbf{a.b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

6. The line with equation $\mathbf{r} = (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \lambda(4\mathbf{i} - \mathbf{j} - \mathbf{k})$ has direction vector $4\mathbf{i} - \mathbf{j} - \mathbf{k}$

The line with equation $\mathbf{r} = (5\mathbf{i} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j})$ has direction vector $\mathbf{i} - \mathbf{j}$

So if the acute angle between the lines is θ, then:

$$\cos\theta = \left|\frac{(4\mathbf{i} - \mathbf{j} - \mathbf{k}).(\mathbf{i} - \mathbf{j})}{|4\mathbf{i} - \mathbf{j} - \mathbf{k}||\mathbf{i} - \mathbf{j}|}\right| \quad \textbf{M1}$$
$$= \left|\frac{4\times 1 + (-1)\times(-1) + (-1)\times 0}{\sqrt{4^2 + (-1)^2 + (-1)^2}\sqrt{1^2 + (-1)^2}}\right| \quad \textbf{M1}$$
$$= \left|\frac{5}{\sqrt{18}\sqrt{2}}\right|$$
$$= \frac{5}{6}$$

So $\theta = \arccos\left(\frac{5}{6}\right) = 33.5573...° = 34°$ to the nearest degree **A1** **[3 Marks]**

Tip: If the line l_1 has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and the line l_2 has equation $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$, then the angle between the lines θ satisfies $\cos\theta = \frac{\mathbf{b.d}}{|\mathbf{b}||\mathbf{d}|}$. Taking the modulus of the right-hand side ensures you get an acute angle since its cosine will be positive.

Alternatively: You can just calculate the angle θ without using the modulus sign. If it happens to be obtuse, then the acute angle between the lines will be $180° - \theta$.

7. The equation of Π will have the form $\mathbf{r.n} = k$ where **n** is a vector that is perpendicular to Π and k satisfies $\mathbf{a.n} = k$ where **a** is the position vector of any particular point in Π

From the question, we know that $\mathbf{n} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}$ and that (5, –2, 3) is a point in Π, so $\mathbf{a} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$

So $k = \mathbf{a.n}$

$$= \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \quad \textbf{M1}$$

$$= 5\times 2 + (-2)\times 6 + 3\times 1 \quad \textbf{M1}$$

$$= 1$$

And so Π has equation $\mathbf{r}.\begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} = 1$ **A1** **[3 Marks]**

Tip: If **n** is a vector that is perpendicular to a plane and **a** is the position vector of any point in the plane, then the scalar product form of the equation of the plane is **r.n** = k where k = **a.n**

8. A normal vector to the plane $\mathbf{r}.(2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}) = 3$ is $2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$
A normal vector to the plane $\mathbf{r}.(8\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 11$ is $8\mathbf{i} - \mathbf{j} + 4\mathbf{k}$
The angle between two planes is the angle between their normal vectors, so if the acute angle between the planes is θ, then:

$$\cos\theta = \left| \frac{(2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}).(8\mathbf{i} - \mathbf{j} + 4\mathbf{k})}{|2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}||8\mathbf{i} - \mathbf{j} + 4\mathbf{k}|} \right| \quad \textbf{M1}$$

$$= \left| \frac{2\times 8 + (-6)\times(-1) + (-3)\times 4}{\sqrt{2^2 + (-6)^2 + (-3)^2}\sqrt{8^2 + (-1)^2 + 4^2}} \right| \quad \textbf{M1}$$

$$= \frac{10}{\sqrt{49}\sqrt{81}}$$

$$= \frac{10}{63}$$

So $\theta = \arccos\left(\frac{10}{63}\right) = 80.8668...^\circ = 80.9^\circ$ (3 s.f.) **A1** **[3 Marks]**

9. The line has equation $\mathbf{r} = (5\mathbf{i} - \mathbf{j} - 3\mathbf{k}) + \lambda(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = \begin{pmatrix} 5-\lambda \\ -1+2\lambda \\ -3+2\lambda \end{pmatrix}$

The plane has equation $\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 12$, so $\mathbf{r}.\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 12$

So the line and plane meet when $\begin{pmatrix} 5-\lambda \\ -1+2\lambda \\ -3+2\lambda \end{pmatrix} . \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 12$ **M1**

$$\therefore (5-\lambda)\times 2 + (-1+2\lambda)\times 3 + (-3+2\lambda)\times(-1) = 12 \quad \textbf{M1}$$

$$10 - 2\lambda - 3 + 6\lambda + 3 - 2\lambda = 12$$

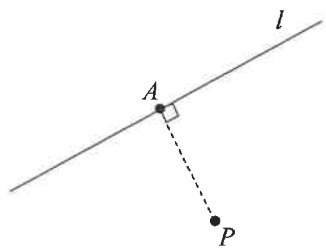
$$2\lambda + 10 = 12$$

$$2\lambda = 2$$

$$\lambda = 1 \quad \textbf{A1}$$

So the line and plane meet when $\lambda = 1$
This corresponds to the point $(5 - 1, -1 + 2\times 1, -3 + 2\times 1) = (4, 1, -1)$ **A1** **[4 Marks]**

10. $l: \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $P = (5, 3, 2)$



The shortest distance between a point and a line is the perpendicular distance between them, as shown to the right

A general point A on l has position vector $\begin{pmatrix} 1+2\lambda \\ 1+\lambda \\ 1+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

The position vector of the point P is $\begin{pmatrix}5\\3\\2\end{pmatrix}$

So $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix}5\\3\\2\end{pmatrix} - \begin{pmatrix}1+2\lambda\\1+\lambda\\1+2\lambda\end{pmatrix} = \begin{pmatrix}4-2\lambda\\2-\lambda\\1-2\lambda\end{pmatrix}$ **M1**

This vector needs to be perpendicular to l, which has direction $\begin{pmatrix}2\\1\\2\end{pmatrix}$, so:

$\begin{pmatrix}4-2\lambda\\2-\lambda\\1-2\lambda\end{pmatrix} \cdot \begin{pmatrix}2\\1\\2\end{pmatrix} = 0$ **M1**

$\therefore (4-2\lambda)\times 2 + (2-\lambda)\times 1 + (1-2\lambda)\times 2 = 12 - 9\lambda = 0$ **M1**

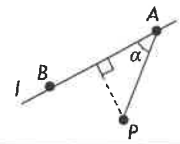
So $\lambda = \frac{12}{9} = \frac{4}{3}$

and so $\overrightarrow{AP} = \begin{pmatrix}4-2\lambda\\2-\lambda\\1-2\lambda\end{pmatrix} = \begin{pmatrix}4-2\times\frac{4}{3}\\2-\frac{4}{3}\\1-2\times\frac{4}{3}\end{pmatrix} = \begin{pmatrix}\frac{4}{3}\\\frac{2}{3}\\-\frac{5}{3}\end{pmatrix}$ **A1**

Hence $\left|\overrightarrow{AP}\right| = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{5}{3}\right)^2} = \sqrt{5}$

So the shortest distance from l to P is $\sqrt{5}$ **A1** **[5 Marks]**

Alternatively: If A and B are two points on L, then the shortest distance from P to L is $\left|\overrightarrow{AP}\right|\sin\alpha$ where α is the acute angle between $\overrightarrow{AP}$ and L and is given by $\cos\alpha = \left|\frac{\overrightarrow{AB}.\overrightarrow{AP}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AP}\right|}\right|$, as shown below:



TOTAL 38 MARKS

