Section 1: Determinants and inverses

Exercise level 3

1. (i) For two general matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, show that $\det(\mathbf{AB})=\det\mathbf{A}\det\mathbf{B}$.

   (ii) A 2x2 matrix $\mathbf{M}$ represents a transformation T which transforms $(1,0)$ and $(0,1)$ to $(1,1)$ and $(-1,1)$ respectively.

   (a) Find $\mathbf{M}$.

   (b) Given that $\mathbf{M}$ can be written as $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}=1$, find the value of λ, where $\lambda>0$.

   (c) Describe the geometric meaning of T.

2. Let $\mathbf{A}=\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}$, $\lambda\neq 0$.

   (i) Show that $\mathbf{B}^{-1}$ exists.

   (ii) Given that $\mathbf{B}^{-1}\mathbf{AB}=\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, find λ, a and b.

   (iii) Show that $(\mathbf{B}^{-1}\mathbf{AB})^2=\mathbf{B}^{-1}\mathbf{A}^2\mathbf{B}$.

   (iv) Find $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{100}$ and hence find $\mathbf{A}^{100}$.

3. (i) Find two values of α such that $\det\begin{pmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{pmatrix}=0$.

   (ii) Let α_1 and α_2 be the values obtained in (i), where $\alpha_1<\alpha_2$.
   Find θ_1 and θ_2 such that

$$\begin{pmatrix} -2-\alpha_1 & \sqrt{3} \\ \sqrt{3} & -\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0\le\theta_1\le\pi$$

$$\begin{pmatrix} -2-\alpha_2 & \sqrt{3} \\ \sqrt{3} & -\alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0\le\theta_2\le\pi.$$

   (iii) Let $\mathbf{P}=\begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}$. Evaluate $\mathbf{P}^2$ and hence find $\mathbf{P}^n$ for the cases when n is even and n is odd.