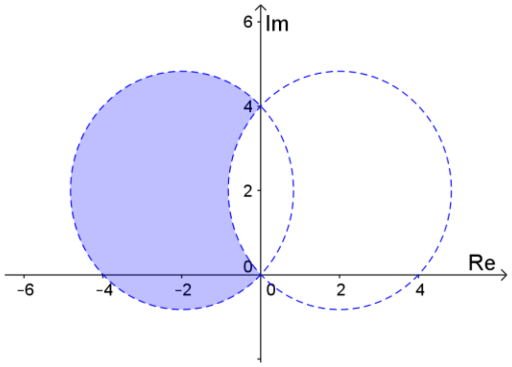
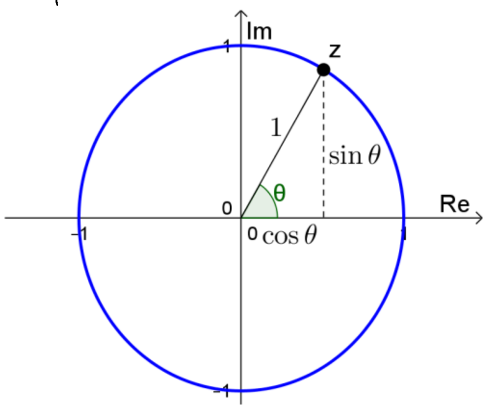
Section 2: Loci in the complex plane

Solutions to Exercise level 3

1. The first inequality represents the inside of a circle with centre -2 + 2i and radius $2\sqrt{2}$.
   The second inequality represents the outside of a circle with centre 2 + 2i and radius $2\sqrt{2}$.

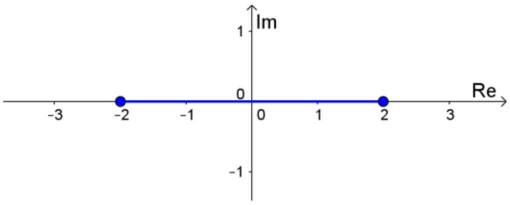


2. (i) $z = \cos\theta + i\sin\theta$
   $= x + iy$
   Since $\sin^2\theta + \cos^2\theta = 1$, then $x^2 + y^2 = 1$
   and therefore z lies on a circle centre the origin with radius 1.



   As θ varies from 0 to 2π, z travels along the full circle.

   (ii) $z = \cos\theta + i\sin\theta + \dfrac{1}{\cos\theta + i\sin\theta}$
   $= \cos\theta + i\sin\theta + \dfrac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$
   $= \cos\theta + i\sin\theta + \dfrac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$
   $= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$
   $= 2\cos\theta$
   As θ varies from 0 to 2π, z travels along the real axis from 2 to -2 and then back to 2. Im(z) = 0.



(iii) $z = \cos\theta + i\sin\theta + \dfrac{1}{(\cos\theta + i\sin\theta)^*} = \cos\theta + i\sin\theta + \dfrac{1}{\cos\theta - i\sin\theta}$

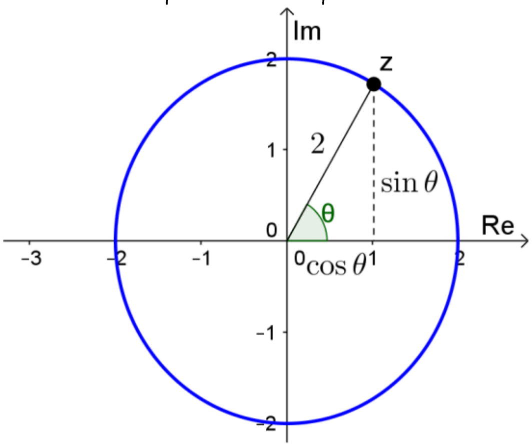
$= \cos\theta + i\sin\theta + \dfrac{\cos\theta + i\sin\theta}{(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)}$

$= \cos\theta + i\sin\theta + \dfrac{\cos\theta + i\sin\theta}{\cos^2\theta + \sin^2\theta}$

$= \cos\theta + i\sin\theta + \cos\theta + i\sin\theta$

$= 2\cos\theta + 2i\sin\theta$

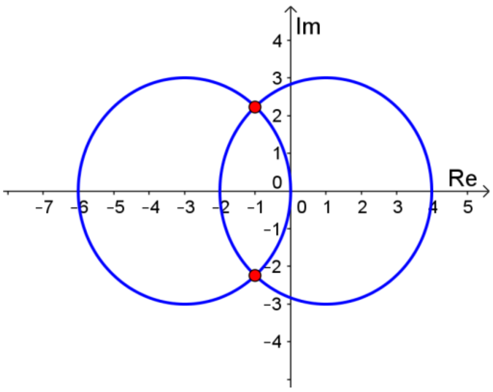
This is similar to part (i) except that in this case the circle has radius 2.



3. (i) $|z-1| = 3$ is a circle, centre (1, 0) with radius 3.

$|z+3| = 3$ is a circle, centre (-3, 0) with radius 3.

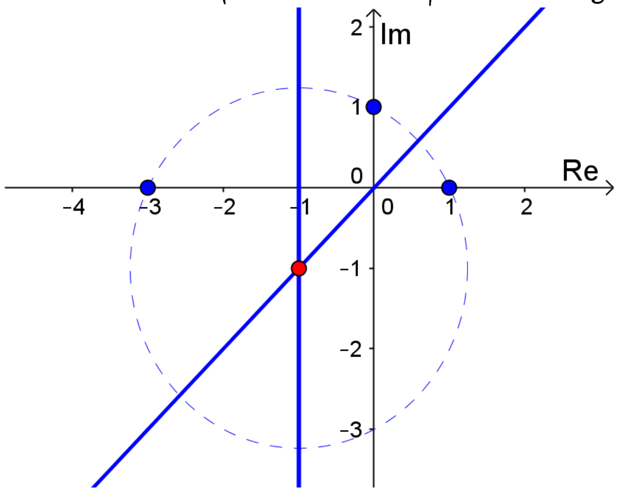
$|z-1| = |z+3| = 3$ is the intersections of these two circles.



(ii) $|z-1| = |z+3| = |z-i|$ is the set of all points that are equidistant from (1, 0), (-3, 0) and (0, 1).

$|z-1| = |z-i|$ is the perpendicular bisector of (1, 0) and (0, 1).

$|z-1|=|z+3|$ is the perpendicular bisector of (1, 0) and (-3, 0).
The only point equidistant from all three points is the intersection of these two perpendicular bisectors. (This point is also the circumcentre of the three points, i.e. the centre of the circle that passes through the three points).



4. (i) If $\arg(z)=\frac{\pi}{4}$, then z must be in the first quadrant, so $a>0$ and $b>0$.

$\arg(z)=\frac{\pi}{4} \Rightarrow \tan(\arg z)=1$, so $\frac{b}{a}=1 \Rightarrow a=b$

(ii) $$\frac{z-2}{z}=\frac{x+iy-2}{x+iy}=\frac{(x-2+iy)(x-iy)}{(x+iy)(x-iy)}$$
$$=\frac{x^2-2x+ixy-ixy+2iy+y^2}{x^2+y^2}$$
$$=\frac{x^2-2x+y^2+2iy}{x^2+y^2}=\frac{x^2-2x+y^2}{x^2+y^2}+\frac{2y}{x^2+y^2}i$$

(iii) For $\arg\left(\frac{z-2}{z}\right)=\frac{\pi}{4}$, from (i) the following conditions must be satisfied:

$x^2-2x+y^2>0$, $2y>0$ and $x^2-2x+y^2=2y$

Considering $x^2-2x+y^2=2y \Rightarrow x^2-2x+y^2-2y=0$
$$\Rightarrow (x-1)^2-1+(y-1)^2-1=0$$
$$\Rightarrow (x-1)^2+(y-1)^2=2$$

This is the equation of a circle, centre (1, 1) and radius $\sqrt{2}$.

The condition $2y>0$ means that the circle is restricted to the part of the circle above the real axis.

The condition $x^2 - 2x + y^2 > 0 \Rightarrow (x-1)^2 - 1 + y^2 > 0$

$$\Rightarrow (x-1)^2 + y^2 > 1$$

This is the outside of a circle with centre (1, 0) and radius 1, so only points outside this circle will satisfy the condition. This is shown in red on the Argand diagram below.