Section 3: Solving systems of differential equations

Exercise level 2 solutions

1. (i) Rearrange the equation $\frac{dx}{dt} = -5x + 6y$ and differentiate

$6y = \frac{dx}{dt} + 5x$ gives $6\frac{dy}{dt} = \frac{d^2x}{dt^2} + 5\frac{dx}{dt}$

Substitute for $\frac{dy}{dt}$ using $\frac{dy}{dt} = -3x + y$

$6(-3x + y) = \frac{d^2x}{dt^2} + 5\frac{dx}{dt}$

Substitute for y using $6y = \frac{dx}{dt} + 5x$

$-18x + \left(\frac{dx}{dt} + 5x\right) = \frac{d^2x}{dt^2} + 5\frac{dx}{dt}$

Rearrange to give

$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 0$

(ii) Auxiliary equation $\lambda^2 + 4\lambda + 13 = 0$

Roots are $\lambda = -2 \pm 3i$

So the general solution for x is $x = e^{-2t}(A\cos 3t + B\sin 3t)$

(iii) As $t \to \infty$ $x \to 0$, so this model predicts that the population is not sustainable.

2. (i) Rearrange the equation $\frac{dy}{dt} = -0.01x + 0.01y$ and differentiate

$0.01x = 0.01y - \frac{dy}{dt}$ gives $0.01\frac{dx}{dt} = 0.01\frac{dy}{dt} - \frac{d^2y}{dt^2}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = 0.02x - 0.12y$

$0.01(0.02x - 0.12y) = 0.01\frac{dy}{dt} - \frac{d^2y}{dt^2}$

Substitute for x using $0.01x = 0.01y - \frac{dy}{dt}$

$0.02\left(0.01y - \frac{dy}{dt}\right) - 0.0012y = 0.01\frac{dy}{dt} - \frac{d^2y}{dt^2}$

Rearrange to give

$\frac{d^2y}{dt^2} - 0.03\frac{dy}{dt} - 0.001y = 0$

(ii) Auxiliary equation $\lambda^2 - 0.03\lambda - 0.001 = 0$

Giving $\lambda = 0.05, -0.02$

So $y = Ae^{0.05t} + Be^{-0.02t}$

(iii) $0.01x = 0.01y - \frac{dy}{dt} = \left(0.01Ae^{0.05t} + 0.01Be^{-0.02t}\right) - \left(0.05Ae^{0.05t} - 0.02Be^{-0.02t}\right)$

$x = \left(-4Ae^{0.05t} + 3Be^{-0.02t}\right)$

When $t = 0$, $x = 700$, $y = 700$

Giving $700 = -4A + 3B$

$700 = A + B$

Solving simultaneously gives $A = 200, B = 500$

So $x = \left(-800e^{0.05t} + 1500e^{-0.02t}\right)$ and $y = 200e^{0.05t} + 500e^{-0.02t}$

(iv) y is always positive

Try x=0

$x = \left(-800e^{0.05t} + 1500e^{-0.02t}\right) = 0$

$1500e^{-0.02t} = 800e^{0.05t}$

$\frac{1500}{800} = \frac{e^{0.05t}}{e^{-0.02t}}$

$e^{0.07t} = \frac{1500}{800}$

$0.07t = \ln\left(\frac{15}{8}\right)$

$t = 8.98$ years

so species A will be extinct in about 9 years.

3. (i) Rearrange the equation $\frac{dy}{dt} = 3x - y$ and differentiate

$3x = \frac{dy}{dt} + y$ gives $3\frac{dx}{dt} = \frac{d^2y}{dt^2} + \frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = 5x - 6y$

$3(5x - 6y) = \frac{d^2y}{dt^2} + \frac{dy}{dt}$

Substitute for x using $3x = \frac{dy}{dt} + y$

$5\left(\frac{dy}{dt} + y\right) - 18y = \frac{d^2y}{dt^2} + \frac{dy}{dt}$

Rearrange to give

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 13y = 0$

Auxiliary equation $\lambda^2 - 4\lambda + 13 = 0$

Roots are $\lambda = 2 \pm 3i$

So the general solution for y is $y = e^{2t}\left(A\cos 3t + B\sin 3t\right)$

Product rule gives $\dfrac{dy}{dt} = 2e^{2t}\left(A\cos 3t + B\sin 3t\right) + e^{2t}\left(-3A\sin 3t + 3B\cos 3t\right)$

Substitute for y and $\dfrac{dy}{dt}$ in the equation $3x = \dfrac{dy}{dt} + y$

$$3x = 2e^{2t}\left(A\cos 3t + B\sin 3t\right) + e^{2t}\left(-3A\sin 3t + 3B\cos 3t\right)$$
$$+ e^{2t}\left(A\cos 3t + B\sin 3t\right)$$

Which gives $x = e^{2t}\left((A+B)\cos 3t + (B-A)\sin 3t\right)$

(ii) When $t = 0$, $x = 1.5$, $y = 0$

$1.5 = A + B$

$0 = A$

Giving $B = 1.5$

So the particular solution is $x = e^{2t}\left(1.5\cos 3t + 1.5\sin 3t\right)$

$y = 1.5e^{2t}\sin 3t$

Which gives the displacement vector $r = \begin{pmatrix} e^{2t}\left(1.5\cos 3t + 1.5\sin 3t\right) \\ 1.5e^{2t}\sin 3t \end{pmatrix}$

4. (i) Rearrange the equation $\dfrac{dy}{dt} = 2x - 5y + b$ and differentiate

$2x = \dfrac{dy}{dt} + 5y - b$ gives $2\dfrac{dx}{dt} = \dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt}$

Substitute for $\dfrac{dx}{dt}$ using $\dfrac{dx}{dt} = x - 4y + a$

$$2\left(x - 4y + a\right) = \frac{d^2y}{dt^2} + 5\frac{dy}{dt}$$

Substitute for x using $x = \dfrac{dy}{dt} + 5y - b$

$$\left(\frac{dy}{dt} + 5y - b\right) - 8y + 2a = \frac{d^2y}{dt^2} + 5\frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2a - b$$

Particular integral, try $y = c$

So $3c = 2a - b$

Auxiliary equation $\lambda^2 - 16 = 0$ has roots $\lambda = \pm 4$

So $y = Ae^{-t} + Be^{-3t} + \dfrac{2a}{3} - \dfrac{b}{3}$

Edexcel FM 2nd order DEs 3 Exercise

Substitute for y and $\frac{dy}{dt}$ in $x = \frac{1}{2}\left(\frac{dy}{dt} + 5y - b\right)$

$$x = \frac{1}{2}\left(\frac{dy}{dt} + 5y - b\right) = \frac{1}{2}\left(\left(-Ae^{-t} - 3Be^{-3t}\right) + 5\left(Ae^{-t} + Be^{-3t} + \frac{2a}{3} - \frac{b}{3}\right) - b\right)$$

$$x = 2Ae^{-t} + Be^{-3t} + \frac{5a}{3} - \frac{4b}{3}$$

Initial conditions $t = 0,\ x = 0,\ y = 0$

$$0 = A + B + \frac{2a}{3} - \frac{b}{3}$$

$$0 = 2A + B + \frac{5a}{3} - \frac{4b}{3}$$

Subtracting $A = b - a$,

Substituting $B = -b + a - \frac{2a}{3} + \frac{b}{3} = \frac{a}{3} - \frac{2b}{3}$

So the solution is $x = \left(2(b-a)e^{-t} + \left(\frac{a}{3} - \frac{2b}{3}\right)e^{-4t} + \frac{5a}{3} - \frac{4b}{3}\right)$,

$$y = (b-a)e^{-t} + \left(\frac{a}{3} - \frac{2b}{3}\right)e^{-4t} + \frac{2a}{3} - \frac{b}{3}$$

(ii) As $t \to \infty$ $x \to \left(\frac{5a}{3} - \frac{4b}{3}\right)$, and $y \to \frac{2a}{3} - \frac{b}{3}$