# Section 3: Solving systems of differential equations

## Exercise level 1 solutions

1. (i) Rearrange the equation $\frac{dy}{dt} = x + 3y$ and differentiate

$x = \frac{dy}{dt} - 3y$ gives $\frac{dx}{dt} = \frac{d^2y}{dt^2} - 3\frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = 2x - 9y$

$$(2x - 9y) = \frac{d^2y}{dt^2} - 3\frac{dy}{dt}$$

Substitute for x using $x = \frac{dy}{dt} - 3y$

$$2\left(\frac{dy}{dt} - 3y\right) - 9y = \frac{d^2y}{dt^2} - 3\frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 15y = 0$$

(ii) Rearrange the equation $\frac{dy}{dt} = 5x + 4y$ and differentiate

$5x = \frac{dy}{dt} - 4y$ gives $5\frac{dx}{dt} = \frac{d^2y}{dt^2} - 4\frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = 2x + 3y$

$$5(2x + 3y) = \frac{d^2y}{dt^2} - 4\frac{dy}{dt}$$

Substitute for x using $5x = \frac{dy}{dt} - 4y$

$$2\left(\frac{dy}{dt} - 4y\right) + 15y = \frac{d^2y}{dt^2} - 4\frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} - 7y = 0$$

(iii) Rearrange the equation $\frac{dy}{dt} = 2x - 4y + 1$ and differentiate

$2x = \frac{dy}{dt} + 4y - 1$ gives $2\frac{dx}{dt} = \frac{d^2y}{dt^2} + 4\frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = x - y$

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$$2(x-y)=\frac{d^2y}{dt^2}+4\frac{dy}{dt}$$

Substitute for x using $2x=\frac{dy}{dt}+4y-1$

$$\left(\frac{dy}{dt}+4y-1\right)-2y=\frac{d^2y}{dt^2}+4\frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}-2y+1=0$$

2. (i) Rearrange the equation $\frac{dx}{dt}=-2x+y$ and differentiate

$y=\frac{dx}{dt}+2x$ gives $\frac{dy}{dt}=\frac{d^2x}{dt^2}+2\frac{dx}{dt}$

Substitute for $\frac{dy}{dt}$ using $\frac{dy}{dt}=x+y$

$$(x+y)=\frac{d^2x}{dt^2}+2\frac{dx}{dt}$$

Substitute for y using $y=\frac{dx}{dt}+2x$

$$x+\left(\frac{dx}{dt}+2x\right)=\frac{d^2x}{dt^2}+2\frac{dx}{dt}$$

Rearrange to give

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}-3x=0$$

(ii) Rearrange the equation $\frac{dx}{dt}=0.1x-0.9y$ and differentiate

$0.9y=0.1x-\frac{dx}{dt}$ gives $0.9\frac{dy}{dt}=0.1\frac{dx}{dt}-\frac{d^2x}{dt^2}$

Substitute for $\frac{dy}{dt}$ using $\frac{dy}{dt}=0.2x+0.3y$

$$0.9(0.2x+0.3y)=0.1\frac{dx}{dt}-\frac{d^2x}{dt^2}$$

Substitute for y using $0.9y=0.1x-\frac{dx}{dt}$

$$0.18x+0.3\left(0.1x-\frac{dx}{dt}\right)=0.1\frac{dx}{dt}-\frac{d^2x}{dt^2}$$

Rearrange to give

$$\frac{d^2x}{dt^2}-0.4\frac{dx}{dt}+0.21x=0$$

(iii) Rearrange the equation $\frac{dx}{dt} = 2x - y + 5$ and differentiate

$y = 2x + 5 - \frac{dx}{dt}$ gives $\frac{dy}{dt} = 2\frac{dx}{dt} - \frac{d^2x}{dt^2}$

Substitute for $\frac{dy}{dt}$ using $\frac{dy}{dt} = 7x - y + 2$

$$\left(7x - y + 2\right) = 2\frac{dx}{dt} - \frac{d^2x}{dt^2}$$

Substitute for y using $y = 2x + 5 - \frac{dx}{dt}$

$$7x - \left(2x + 5 - \frac{dx}{dt}\right) + 2 = 2\frac{dx}{dt} - \frac{d^2x}{dt^2}$$

Rearrange to give

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} + 5x - 3 = 0$$

3. (i) Rearrange the equation $\frac{dy}{dt} = x + y$ and differentiate

$x = \frac{dy}{dt} - y$ gives $\frac{dx}{dt} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = 2x + 6y$

$$\left(2x + 6y\right) = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Substitute for x using $x = \frac{dy}{dt} - y$

$$2\left(\frac{dy}{dt} - y\right) + 6y = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$$

Auxiliary equation $\lambda^2 - 3\lambda - 4 = 0$

Which factorises to $(\lambda - 4)(\lambda + 1) = 0$

$\lambda^2 - 3\lambda - 4 = 0$ Roots are $\lambda = 4, -1$

So the general solution for y is $y = Ae^{4t} + Be^{-t}$

Substitute for y and $\frac{dy}{dt}$ in the equation $x = \frac{dy}{dt} - y$

Which gives $x = \left(4Ae^{4t} + (-1)Be^{-t}\right) - \left(Ae^{4t} + Be^{-t}\right) = 3Ae^{4t} - 2Be^{-t}$

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(ii) Rearrange the equation $\frac{dx}{dt} = x + 4y$ and differentiate

$4y = \frac{dx}{dt} - x$ gives $4\frac{dy}{dt} = \frac{d^2x}{dt^2} - \frac{dx}{dt}$

Substitute for $\frac{dy}{dt}$ using $\frac{dy}{dt} = 2x - y$

$4(2x - y) = \frac{d^2x}{dt^2} - \frac{dx}{dt}$

Substitute for y using $4y = \frac{dx}{dt} - x$

$8x - \left(\frac{dx}{dt} - x\right) = \frac{d^2x}{dt^2} - \frac{dx}{dt}$

Rearrange to give

$\frac{d^2x}{dt^2} + 0\frac{dx}{dt} - 9x = 0$

$\frac{d^2x}{dt^2} - 9x = 0$

Auxiliary equation $\lambda^2 - 9 = 0$

Roots are $\lambda = \pm 3$

So the general solution for x is $x = Ae^{3t} + Be^{-3t}$

Substitute for x and $\frac{dx}{dt}$ in the equation $y = \frac{1}{4}\left(\frac{dx}{dt} - x\right)$

Which gives $y = \frac{1}{4}\left(\left(3Ae^{3t} + (-3)Be^{-3t}\right) - \left(Ae^{3t} + Be^{-3t}\right)\right) = \frac{1}{4}\left(2Ae^{3t} - 4Be^{-3t}\right)$

$y = 0.5Ae^{3t} - Be^{-3t}$

(iii) Rearrange the equation $\frac{dy}{dt} = 2x - 9y + 7$ and differentiate

$2x = \frac{dy}{dt} + 9y - 7$ gives $2\frac{dx}{dt} = \frac{d^2y}{dt^2} + 9\frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt} = x - 13y - 5$

$2(x - 13y - 5) = \frac{d^2y}{dt^2} + 9\frac{dy}{dt}$

Substitute for x using $x = \frac{dy}{dt} + 9y - 7$

$\left(\frac{dy}{dt} + 9y - 7\right) - 26y - 10 = \frac{d^2y}{dt^2} + 9\frac{dy}{dt}$

Rearrange to give

$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 17y = -17$

Auxiliary equation $\lambda^2 + 8\lambda + 17 = 0$

Roots are $\lambda = -4 \pm i$

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Particular integral:

Try $y=c$

$$\frac{d^2y}{dt^2}+8\frac{dy}{dt}+17y=0+0+17c=-17 \text{ giving } y=-1$$

So the general solution for y is $y=e^{-4t}(A\cos t+B\sin t)-1$

Product rule gives $\frac{dy}{dt}=(-4)e^{-4t}(A\cos t+B\sin t)+e^{-4t}(-A\sin t+B\cos t)$

Substitute for y and $\frac{dy}{dt}$ in the equation

$$x=e^{-4t}((-4A+B)\cos t+(-4B-A)\sin t)+9\left(e^{-4t}(A ocs t+B\sin t)-1\right)-7$$

Which gives $x=e^{-4t}((5A+B)\cos t+(5B-A)\sin t)-16$

4. (i) Rearrange the equation $\frac{dy}{dt}=x-3y$ and differentiate

$x=\frac{dy}{dt}+3y$ gives $\frac{dx}{dt}=\frac{d^2y}{dt^2}+3\frac{dy}{dt}$

Substitute for $\frac{dx}{dt}$ using $\frac{dx}{dt}=3x+7y$

$$(3x+7y)=\frac{d^2y}{dt^2}+3\frac{dy}{dt}$$

Substitute for x using $x=\frac{dy}{dt}+3y$

$$3\left(\frac{dy}{dt}+3y\right)+7y=\frac{d^2y}{dt^2}+3\frac{dy}{dt}$$

Rearrange to give

$$\frac{d^2y}{dt^2}-16y=0$$

Auxiliary equation $\lambda^2-16=0$ has roots $\lambda=\pm4$

So $y=Ae^{4t}+Be^{-4t}$

Substitute for $y$ and $\frac{dy}{dt}$ in $x=\frac{dy}{dt}+3y$

$$x=\left(4Ae^{4t}+(-4)Be^{-4t}\right)+3\left(Ae^{4t}+Be^{-4t}\right)=\left(7Ae^{4t}-Be^{-4t}\right)$$

Initial conditions $t=0,\ x=0,\ y=16$

$10=A+B$

$0=(7A-B)$

Giving $A=2, B=14$

So the solution is $x=\left(14e^{4t}-14e^{-4t}\right),\ y=2e^{4t}+14e^{-4t}$