Section 2: Second order non-homogeneous equations

Exercise level 2 solutions

1. (i) Resultant force is $5mx - 4mv + 2m$

Newton's second law $5mx - 4mv + 2m = ma$

$$5x - 4\frac{dx}{dt} + 2 = \frac{d^2x}{dt^2}$$

which rearranges to give $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} - 5x = 2$

(ii) Auxiliary equation $\lambda^2 + 4\lambda - 5 = 0$

$$(\lambda + 5)(\lambda - 1) = 0$$

Roots $\lambda = -5, 1$

So the complementary function is $x = Ae^t + Be^{-5t}$

Particular integral try $x = k$

$$\frac{dx}{dt} = 0, \frac{d^2x}{dt^2} = 0$$

Substitute into the DE $\quad -5k = 2$ so $k = -0.4$

The general solution is $x = Ae^t + Be^{-5t} - 0.4$

(iii) Using the initial conditions $t = 0, x = 0.6$ gives $0.6 = A + B - 0.4$ giving $A + B = 1$

Differentiate general solution to give $\frac{dx}{dt} = Ae^t - 5Be^{-5t}$

Using the initial conditions $t = 0, \frac{dx}{dt} = 0$ gives $0 = A - 5B$

Solving simultaneously gives $A = \frac{5}{6}, B = \frac{1}{6}$

So $x = \frac{5}{6}e^t + \frac{1}{6}e^{-5t} - 0.4$

2. (i) Auxiliary equation is $3\lambda^2 + 4\lambda + 1 = 0$

$(3\lambda + 1)(\lambda + 1) = 0$ So roots are $\lambda = -\frac{1}{3}, -1$

So the complementary function is $x = Ae^{-\frac{1}{3}t} + Be^{-t}$

(ii) To find the particular integral try $x = a\cos t + b\sin t$

Differentiate $\frac{dx}{dt} = -a\sin t + b\cos t$ and $\frac{d^2x}{dt^2} = -a\cos t - b\sin t$

Substituting

$3(-a\cos t - b\sin t) + 4(-a\sin t + b\cos t) + a\cos t + b\sin t = \cos t$

Equate coefficients of $\cos t$ $\; -3a + 4b + a = 1$ giving $-2a + 4b = 1$

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Equate coefficients of $\sin t$ $-3b-4a+b=0$ giving $-4a-2b=0$
Solving simultaneously (calculator method) $a=-0.1, b=0.2$

So the general solution of the DE is $x = Ae^{-\frac{1}{3}t} + Be^{-t} - 0.1\cos t + 0.2\sin t$

(iii) Initial conditions give specific values for A and B

As $t \to \infty$, $Ae^{-\frac{1}{3}t} \to 0$ for all values of A and $Be^{-t} \to 0$ for all values of B
Hence for large values of t, $x \approx -0.1\cos t + 0.2\sin t$ which is independent of the initial conditions.

3. (i) Auxiliary equation is $\lambda^2 + 2\lambda + 10 = 0$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4\times 10}}{2} = -1 \pm 3i$$

So the complementary function is $x = e^{-t}(A\cos 3t + B\sin 3t)$
For the particular integral, try $x = at^2 + bt + c$

Differentiate to give $\frac{dx}{dt} = 2at + b$ and $\frac{d^2x}{dt^2} = 2a$

Substituting into the DE gives $2a + 2(2at+b) + 10(at^2+bt+c) = t^2$

Equating the coefficients of t^2 gives $10a = 1$ so $a = \frac{1}{10}$

Equating the coefficients of t gives $4a + 10b = 0$ so $b = \frac{1}{10}\left(-4\times\frac{1}{10}\right) = -\frac{1}{25}$

Equating constant term gives $2a + 2b + 10c = 0$ so $c = -\frac{3}{250}$

So the general solution is $x = e^{-t}(A\cos 3t + B\sin 3t) + \frac{t^2}{10} - \frac{t}{25} - \frac{3}{250}$

(ii) Initial conditions, when $t=0, x=0$ giving

$0 = 1(A + B\times 0) - \frac{3}{250}$ giving $A = \frac{3}{250}$

Differentiate to give

$$\frac{dx}{dt} = -e^{-t}(A\cos 3t + B\sin 3t) + e^{-t}(-3A\sin 3t + 3Bsos3t) + \frac{t}{5} - \frac{1}{25}$$

When $t=0, \frac{dx}{dt} = 0$

$0 = -A + 3B - \frac{1}{25}$ giving $B = \frac{1}{3}\left(\frac{1}{25} + \frac{3}{250}\right) = \frac{13}{750}$

$$x = e^{-t}\left(\frac{3}{250}\cos 3t + \frac{13}{750}\sin 3t\right) + \frac{t^2}{10} - \frac{t}{25} - \frac{3}{250}$$

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4. (i) Auxiliary equation $\lambda^2 + k^2 = 0$ has roots $\lambda = \pm ki$

Which gives $\theta = A\cos kt + B\sin kt$

(ii) Particular integral might be $\theta = a\cos kt + b\sin kt$

but this is also the complementary function, so try $\theta = t(a\cos kt + b\sin kt)$

Differentiate $\dfrac{d\theta}{dt} = (a\cos kt + b\sin kt) + t(-ak\sin kt + bk\cos kt)$

$$\frac{d^2\theta}{dt^2} = (-ak\sin kt + bk\cos kt) + (-ak\sin kt + bk\cos kt) + t\left(-ak^2\cos kt - bk^2\sin kt\right)$$

Substitute into the DE

$$\left(2(-ak\sin kt + bk\cos kt) + t\left(-ak^2\cos kt - bk^2\sin kt\right)\right) + k^2t(a\cos kt + b\sin kt) = \cos kt$$

So $2(-ak\sin kt + bk\cos kt) = \cos kt$

Equate coefficient of $\cos t$ $\;2bk = 1$ so $b = \dfrac{1}{2k}$

Equate coefficient of $\sin t$ $\;-2ak = 0$ so $a = 0$

So the general solution is $\theta = A\cos kt + B\sin kt + \dfrac{1}{2k}t\sin kt$

When $t = 0, \theta = \alpha$ which gives $\alpha = A + B\times 0 + 0$ so $A = \alpha$

$$\frac{d\theta}{dt} = -Ak\sin kt + Bk\cos kt + \frac{1}{2k}(\sin kt + tk\cos kt)$$

When $t = 0, \dfrac{d\theta}{dt} = 0$ so $Bk = 0$ giving $B = 0$

So the solution of the DE is $\theta = \alpha\cos kt + \dfrac{1}{2k}t\sin kt$

(iii) As t increases, the second term gives oscillations of increasing amplitude, so the model breaks down. The model is only good for small values of θ so the model is not good.

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