Section 2: Non-homogeneous differential equations

Solutions to Exercise level 1

1. (i) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x^2$

Auxiliary equation: $k^2 - 3k + 2 = 0$

$(k-1)(k-2) = 0$

$k = 1$ or $k = 2$

Complementary function is $y = Ae^x + Be^{2x}$

Particular integral is $y = ax^2 + bx + c$

$\dfrac{dy}{dx} = 2ax + b$

$\dfrac{d^2y}{dx^2} = 2a$

Substituting into differential equation:

$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 4x^2$

$2ax^2 + (-6a + 2b)x + 2a - 3b + 2c = 4x^2$

Equating coefficients of x^2: $2a = 4 \Rightarrow x = 2$

Equating coefficients of x: $-6a + 2b = 0 \Rightarrow b = 3a = 6$

Equating constant terms: $2a - 3b + 2c = 0 \Rightarrow 2c = 3b - 2a = 14$

$\Rightarrow c = 7$

General solution is $y = Ae^x + Be^{2x} + 2x^2 + 6x + 7$

(ii) $\dfrac{d^2y}{dx^2} + 4y = e^{2x}$

Auxiliary equation: $k^2 + 4 = 0$

$k^2 = -4$

$k = \pm 2i$

Complementary function is $y = A\cos 2x + B\sin 2x$

Particular integral is $y = ae^{2x}$

$\dfrac{dy}{dx} = 2ae^{2x}$

$\dfrac{d^2y}{dx^2} = 4ae^{2x}$

Substituting into differential equation: $4ae^{2x}+4ae^{2x}=e^{2x}$

$8a=1$

$a=\frac{1}{8}$

General solution is $y=A\cos 2x+B\sin 2x+\frac{1}{8}e^{2x}$

(iii) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\sin 2x$

Auxiliary equation: $k^2+2k+1=0$

$(k+1)^2=0$

$k=-1$

Complementary function is $y=(Ax+B)e^{-x}$

Particular integral is $y=a\sin 2x+b\cos 2x$

$$\frac{dy}{dx}=2a\cos 2x-2b\sin 2x$$

$$\frac{d^2y}{dx^2}=-4a\sin 2x-4b\cos 2x$$

Substituting into differential equation:

$$-4a\sin 2x-4b\cos 2x+2(2a\cos 2x-2b\sin 2x)$$
$$+a\sin 2x+b\cos 2x=\sin 2x$$
$$(-3a-4b)\sin 2x+(4a-3b)\cos 2x=\sin 2x$$

Comparing coefficients of cos 2x: $4a-3b=0 \quad \Rightarrow b=\frac{4}{3}a$

Comparing coefficients of sin 2x: $-3a-4b=1 \quad \Rightarrow -3a-\frac{16}{3}a=1$

$\Rightarrow -\frac{25}{3}a=1$

$\Rightarrow a=-\frac{3}{25}, b=-\frac{4}{25}$

General solution is $y=(Ax+B)e^{-x}-\frac{3}{25}\sin 2x-\frac{4}{25}\cos 2x$

(iv) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=6e^{x}$

Auxiliary equation: $k^2+k-2=0$

$(k-1)(k+2)=0$

$k=1$ or $k=-2$

Complementary function is $y=Ae^{x}+Be^{-2x}$

The RHS of the differential equation has the same form as part of the complementary function.

So the particular integral is $y = axe^x$

$$\frac{dy}{dx} = ae^x + axe^x$$

$$\frac{d^2y}{dx^2} = ae^x + ae^x + axe^x = 2ae^x + axe^x$$

Substituting into the differential equation:

$$2ae^x + axe^x + ae^x + axe^x - 2axe^x = 6e^x$$

$$3a = 6$$

$$a = 2$$

General solution is $y = Ae^x + Be^{-2x} + 2xe^x$

(v) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 10y = x + \cos x$

Auxiliary equation: $k^2 - 6k + 10 = 0$

$$k = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 10}}{2} = \frac{6 \pm \sqrt{-4}}{2} = 3 \pm i$$

Complementary function is $y = e^{3x}(A\cos x + B\sin x)$

Particular integral is $y = ax + b + c\cos x + d\sin x$

$$\frac{dy}{dx} = a - c\sin x + d\cos x$$

$$\frac{d^2y}{dx^2} = -c\cos x - d\sin x$$

Substituting into differential equation:

$$-c\cos x - d\sin x - 6(a - c\sin x + d\cos x)$$
$$+10(ax + b + c\cos x + d\sin x) = x + \cos x$$

$$(9c - 6d)\cos x + (9d + 6c)\sin x + 10ax + 10b - 6a = x + \cos x$$

Equating coefficients of x: $10a = 1 \quad \Rightarrow a = \frac{1}{10}$

Equating constant terms: $10b - 6a = 0 \quad \Rightarrow b = \frac{3}{5}a = \frac{3}{50}$

Equating coefficients of sin x: $9d + 6c = 0 \quad \Rightarrow c = -\frac{3}{2}d$

Equating coefficients of cos x: $9c - 6d = 1 \quad \Rightarrow -\frac{27}{2}d - 6d = 1$

$$\Rightarrow -\frac{39}{2}d = 1$$

$$\Rightarrow d = -\frac{2}{39}, c = \frac{1}{13}$$

General solution is $y = e^{3x}(A\cos x + B\sin x) + \frac{1}{10}x + \frac{3}{50} + \frac{1}{13}\cos x - \frac{2}{39}\sin x$

(vi) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

Auxiliary equation: $k^2 + 4k + 4 = 0$

$(k+2)^2 = 0$

$k = -2$

Complementary function is $y = (Ax + B)e^{-2x}$

The complementary function contains terms in both e^{-2x} and xe^{-2x} so particular integral is $y = ax^2e^{-2x}$

$$\frac{dy}{dx} = 2axe^{-2x} - 2ax^2e^{-2x}$$

$$\frac{d^2y}{dx^2} = 2ae^{-2x} - 4axe^{-2x} - 4axe^{-2x} + 4ax^2e^{-2x}$$

$$= 2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x}$$

Substituting into differential equation:

$$2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x} + 4(2axe^{-2x} - 2ax^2e^{-2x}) + 4ax^2e^{-2x} = e^{-2x}$$

$$2ae^{-2x} - 8axe^{-2x} + 4ax^2e^{-2x} + 8axe^{-2x} - 8ax^2e^{-2x} + 4ax^2e^{-2x} = e^{-2x}$$

$$2ae^{-2x} = e^{-2x}$$

$$a = \tfrac{1}{2}$$

General solution is $y = (\tfrac{1}{2}x^2 + Ax + B)e^{-2x}$

2. (i) $\frac{d^2y}{dx^2} + y = 10e^{-2x}$

Auxiliary equation is $k^2 + 1 = 0$

$k^2 = -1$

$k = \pm i$

Complementary function is $y = A\sin x + B\cos x$

Particular integral is $y = ae^{-2x}$

$$\frac{dy}{dx} = -2ae^{-2x}$$

$$\frac{d^2y}{dx^2} = 4ae^{-2x}$$

Substituting into differential equation:

$$4ae^{-2x} + ae^{-2x} = 10e^{-2x}$$

$$a = 2$$

General solution is $y = A\sin x + B\cos x + 2e^{-2x}$

When $x = 0$, $y = 0$: $\quad 0 = B + 2 \quad \Rightarrow B = -2$

$$\frac{dy}{dx} = A\cos x - B\sin x - 4e^{-2x}$$

When $x = 0$, $\frac{dy}{dx} = 0$: $\quad 0 = A - 4 \quad \Rightarrow A = 4$

Particular solution is $y = 4\sin x - 2\cos x + 2e^{-2x}$

(ii) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 50\sin x$

Auxiliary equation is $k^2 - k - 6 = 0$

$$(k-3)(k+2) = 0$$

$$k = 3 \text{ or } k = -2$$

Complementary function is $y = Ae^{3x} + Be^{-2x}$

Particular integral is $y = a\sin x + b\cos x$

$$\frac{dy}{dx} = a\cos x - b\sin x$$

$$\frac{d^2y}{dx^2} = -a\sin x - b\cos x$$

Substituting into differential equation:

$$-a\sin x - b\cos x - (a\cos x - b\sin x) - 6(a\sin x + b\cos x) = 50\sin x$$

$$(-7a + b)\sin x + (-7b - a)\cos x = 50\sin x$$

Equating coefficients of cos x: $\quad -7b - a = 0 \quad \Rightarrow a = -7b$

Equating coefficients of sin x: $\quad -7a + b = 50 \quad \Rightarrow 49b + b = 50$

$$\Rightarrow b = 1, a = -7$$

General solution is $y = Ae^{3x} + Be^{-2x} - 7\sin x + \cos x$

When $x = 0$, $y = 1$: $\quad 1 = A + B + 1 \quad \Rightarrow B = -A$

$$\frac{dy}{dx} = 3Ae^{3x} - 2Be^{-2x} - 7\cos x - \sin x$$

When $x = 0$, $\frac{dy}{dx} = 3$: $\quad 3 = 3A - 2B - 7 \quad \Rightarrow 10 = 3A + 2A$

$$\Rightarrow A = 2, B = -2$$

Particular solution is $y = 2e^{3x} - 2e^{-2x} - 7\sin x + \cos x$

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(iii) $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 2x^2 - 3$

Auxiliary equation is $2k^2 - k - 1 = 0$

$$(2k+1)(k-1) = 0$$

$$k = -\tfrac{1}{2} \text{ or } k = 1$$

Complementary function is $y = Ae^{-\frac{1}{2}x} + Be^{x}$

Particular integral is $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

Substituting into differential equation:

$$2 \times 2a - (2ax + b) - (ax^2 + bx + c) = 2x^2 - 3$$

$$-ax^2 + (-2a - b)x + 4a - b - c = 2x^2 - 3$$

Equating coefficients of x^2: $-a = 2 \quad \Rightarrow a = -2$

Equating coefficients of x: $-2a - b = 0 \quad \Rightarrow b = -2a = 4$

Equating constant terms: $4a - b - c = -3 \quad \Rightarrow -8 - 4 - c = -3 \quad \Rightarrow c = -9$

General solution is $y = Ae^{-\frac{1}{2}x} + Be^{x} - 2x^2 + 4x - 9$

When $x = 0,\ y = 0$: $\quad 0 = A + B - 9 \quad \Rightarrow B = 9 - A$

$$\frac{dy}{dx} = -\tfrac{1}{2}Ae^{-\frac{1}{2}x} + Be^{x} - 4x + 4$$

When $x = 0,\ \frac{dy}{dx} = 4$: $\quad 4 = -\tfrac{1}{2}A + B + 4 \quad \Rightarrow 0 = -\tfrac{1}{2}A + 9 - A$

$$\Rightarrow \tfrac{3}{2}A = 9$$

$$\Rightarrow A = 6, B = 3$$

Particular solution is $y = 6e^{-\frac{1}{2}x} + 3e^{x} - 2x^2 + 4x - 9$