Section 1: Homogeneous differential equations

Exercise level 2 solutions

1. (i) To solve $\frac{d^2x}{dt^2}+0.09x=0$

   Auxiliary equation $\lambda^2+0.09=0$

   (ii) Roots $\lambda=\pm 0.3i$

   So the general solution is $x=A\cos 0.3t+B\sin 0.3t$

   (iii) Differentiate displacement to find velocity

   $v=\frac{dx}{dt}=-0.3A\sin 0.3t+0.3B\cos 0.3t$

   (iv) When $t=0, x=0.1$ giving $0.1=A\cos 0+B\sin 0=A$

   When $t=0, v=0$ giving $0=-0.3A\sin 0+0.3B\cos 0=0.3B$

   So $A=0.1, B=0$

   $x=0.1\cos 0.3t$

2. (i) $\frac{d^2\theta}{dt^2}+6\frac{d\theta}{dt}+11\theta=0$

   Auxiliary equation $\lambda^2+6\lambda+11=0$

   Roots $\lambda=-3\pm\sqrt{2}i$

   So general solution is $\theta=e^{-3t}\left(A\cos\sqrt{2}t+B\sin\sqrt{2}t\right)$

   (ii) When $t=0, \theta=0$ giving $0=1\times(A\cos 0+B\sin 0)=A$ so $A=0$

   Differentiate $\theta=Be^{-3t}\sin\sqrt{2}t$ using the product rule

   $\frac{d\theta}{dt}=-3Be^{-3t}\sin\sqrt{2}t+Be^{-3t}\sqrt{2}\cos\sqrt{2}t$

   When $t=0, \frac{d\theta}{dt}=0.5$ giving $\frac{d\theta}{dt}=0.5=+B\sqrt{2}\cos 0=B\sqrt{2}$ so $=\frac{1}{2\sqrt{2}}$

   So the particular solution is $\theta=\frac{1}{2\sqrt{2}}e^{-3t}\sin\sqrt{2}t$

   (iii) Maximum angle will be the first time at which $\frac{d\theta}{dt}=0$

   $\frac{d\theta}{dt}=-3\frac{1}{2\sqrt{2}}e^{-3t}\sin\sqrt{2}t+\frac{1}{2\sqrt{2}}e^{-3t}\sqrt{2}\cos\sqrt{2}t=0$

$$-3\sin\sqrt{2}t+\sqrt{2}\cos\sqrt{2}t=0$$

$$\frac{\sin\sqrt{2}t}{\cos\sqrt{2}t}=\tan\sqrt{2}t=\frac{\sqrt{2}}{3}$$

So $t=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)\approx 0.31$ to 2 s.f.

(iv) For large values of t $\theta\to 0$ and the pendulum will oscillate with decreasing amplitude tending towards hanging vertically.

3. (i) For $\frac{d^2h}{dx^2}+5\frac{dh}{dt}+6h=0$

Auxiliary equation is $\lambda^2+5\lambda+6=0$

Roots are $\lambda=-2,-3$

So the general solution is $h=Ae^{-2t}+Be^{-3t}$

Given $t=0, h=0$ $0=Ae^{0}+Be^{0}=A+B$

Differentiate for velocity

$$\frac{dh}{dt}=-2Ae^{-2t}-3Be^{-3t}$$

Given $t=0, \frac{dh}{dt}=2$ $\frac{dh}{dt}=2=-2A-3B$

Solving the simultaneous equations

$A=2, B=-2$

So $h=2e^{-2t}-2e^{-3t}$

(ii) Max height when $\frac{dh}{dt}=-2Ae^{-2t}-3Be^{-3t}=0$

$-4e^{-2t}+6e^{-3t}=0$

Divide by e^{-3t} to give $-4e^{t}+6=0$

So $e^{t}=\frac{3}{2}$ giving $t=\ln\left(\frac{3}{2}\right)$

Max height $h_{max}=2e^{-2\ln\left(\frac{3}{2}\right)}-2e^{-3\ln\left(\frac{3}{2}\right)}=2\times\left(\frac{3}{2}\right)^{-2}-2\times\left(\frac{3}{2}\right)^{-3}=\frac{8}{27}$

(iii) As $t\to\infty, h\to 0$ but never reaches 0. The object is likely to reach the bottom of the seabed in finite time, so the model may not be good for large t.

4. (i) $\frac{d^2x}{dt^2}-0.4\frac{dx}{dt}+0.04x=0$

Auxiliary equation $\lambda^2-0.4\lambda+0.04=0$

Repeated root $\lambda = 0.2$

So the general solution is $x = Ae^{0.2t} + Bte^{0.2t}$

(ii) When $t = 0$, $x = 3$ so $3 = A + B \times 0$ so $A = 3$

When $t = 3$, $x = 0$ so $0 = Ae^{0.6} + B \times 3e^{0.6}$ so $B = -1$

Which gives $x = 3e^{0.2t} - te^{0.2t}$

(iii) Differentiate to find velocity

$$v = \frac{dx}{dt} = 3 \times 0.2e^{0.2t} - \left(1 \times e^{0.2t} + 0.2te^{0.2t}\right)$$

$$v = -0.4e^{0.2t} - 0.2te^{0.2t} = 0$$

$$t = -\frac{0.4e^{0.2t}}{0.2e^{0.2t}} = -2$$

So the velocity is never zero for positive values of t.

(iv) As $t \to \infty$, $v \to -\infty$ so the model predicts impossibly high values of velocity for large values of t, so it is not a good model.