Edexcel Further Maths First order DEs

Section 2: Integrating factors

Solutions to Exercise level 2

1. (i) $\dfrac{dy}{dx}+\dfrac{y}{1+x}=x^{\frac{3}{2}}$

Integrating factor $=e^{\int \frac{1}{1+x}dx}=e^{\ln(1+x)}=1+x$

$$(1+x)\frac{dy}{dx}+y=x^{\frac{3}{2}}(1+x)$$

$$\frac{d}{dx}(y(1+x))=x^{\frac{3}{2}}+x^{\frac{5}{2}}$$

$$y(1+x)=\tfrac{2}{5}x^{\frac{5}{2}}+\tfrac{2}{7}x^{\frac{7}{2}}+A$$

$$y=\frac{\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}+A}{1+x}$$

When $x=0$, $y=2$, so $2=A$

Particular solution is $y=\dfrac{\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}+2}{1+x}$

As x tends to infinity, y tends to infinity.

(ii) $(1+x)\dfrac{dy}{dx}+ny=x^{\frac{3}{2}}(1+x)^{2-n}$

$$\frac{dy}{dx}+\frac{ny}{1+x}=x^{\frac{3}{2}}(1+x)^{1-n}$$

Integrating factor $=e^{\int \frac{n}{1+x}dx}=e^{n\ln(1+x)}=(1+x)^n$

$$(1+x)^n\frac{dy}{dx}+n(1+x)^{n-1}y=x^{\frac{3}{2}}(1+x)$$

$$\frac{d}{dx}(y(1+x)^n)=x^{\frac{3}{2}}+x^{\frac{5}{2}}$$

$$y(1+x)^n=\tfrac{2}{5}x^{\frac{5}{2}}+\tfrac{2}{7}x^{\frac{7}{2}}+A$$

$$y=\frac{\frac{2}{5}x^{\frac{5}{2}}+\frac{2}{7}x^{\frac{7}{2}}+A}{(1+x)^n}$$

$$y=\frac{2x^{\frac{5}{2}}(\frac{1}{5}+\frac{1}{7}x)+A}{(1+x)^n}$$

$$y=\frac{2x^{\frac{5}{2}}(7+5x)+C}{35(1+x)^n}$$

replacing A with $\dfrac{C}{35}$

If $n<\frac{7}{2}$, then y tends to infinity as x tends to infinity.

If $n = \frac{7}{2}$, then y tends to $\frac{2}{7}$ as x tends to infinity.

If $n > \frac{7}{2}$, then y tends to 0 as x tends to infinity.

2. (i) $(1+x^2)\frac{dy}{dx} - \frac{4x^3y}{1-x^2} = 1$

$$\frac{dy}{dx} - \frac{4x^3y}{(1-x^2)(1+x^2)} = \frac{1}{1+x^2}$$

Integrating factor $= e^{\int -\frac{4x^3}{1-x^4}dx} = e^{\ln(1-x^4)} = 1-x^4 = (1-x^2)(1+x^2)$

$$(1-x^4)\frac{dy}{dx} - 4x^3y = 1-x^2$$

$$\frac{d}{dx}(y(1-x^4)) = 1-x^2$$

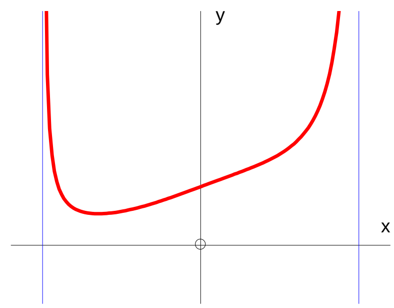
$$y(1-x^4) = x - \tfrac{1}{3}x^3 + A$$

Replacing A with $\frac{k}{3}$

$$y = \frac{3x - x^3 + k}{3(1-x^4)}$$

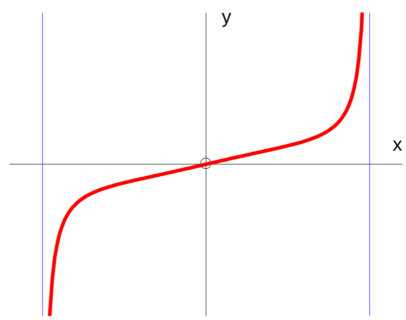
(ii) (A) $y = 1$ when $x = 0$, so $1 = \frac{k}{3} \Rightarrow k = 3$

Particular solution is $y = \dfrac{3x - x^3 + 3}{3(1-x^4)}$



(B) $y = 0$ when $x = 0$, so $0 = \frac{k}{3} \Rightarrow k = 0$

Particular solution is $y = \dfrac{3x - x^3}{3(1-x^4)}$



(iii) The denominator $= (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)$

For there to be a finite limit as x tends to 1, the numerator must have a factor $(1-x)$.

By the factor theorem, $k + 3x - x^3 = 0$ when $x = 1$

$$k + 3 - 1 = 0$$

$$k = -2$$

3. (i) The equation $\frac{di}{dt}+\frac{R}{L}i=\frac{V}{L}$ is linear as R, L and V are constant.

$\int P(t)\,dt=\int\frac{R}{L}dt=\frac{R}{L}t$

Integrating factor is $e^{\int P(t)\,dt}=e^{\frac{R}{L}t}$

So the equation becomes $\frac{dv}{dt}\left(e^{\frac{R}{L}t}i\right)=\frac{V}{L}e^{\frac{R}{L}t}$

$\left(e^{\frac{R}{L}t}i\right)=\int\left(\frac{V}{L}e^{\frac{R}{L}t}\right)dt=\frac{V}{L}\times\frac{L}{R}e^{\frac{R}{L}t}+c=\frac{V}{R}e^{\frac{R}{L}t}+c$

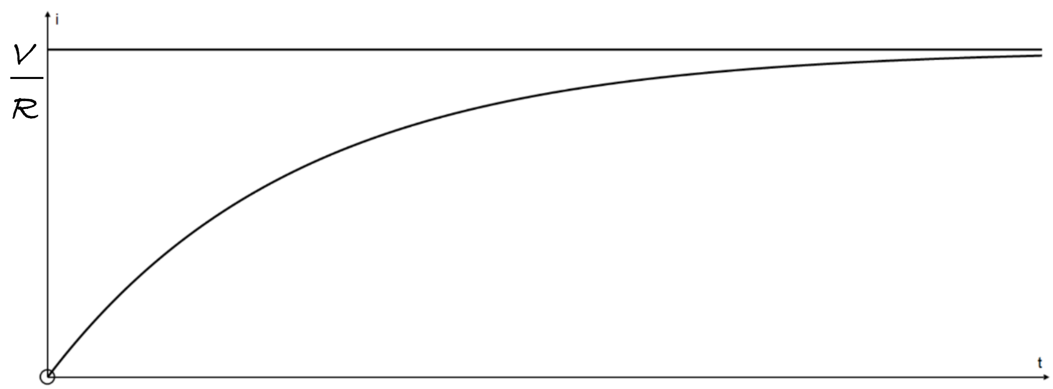
Rearranging $i=\frac{V}{R}+ce^{-\frac{R}{L}t}$

When $t=0, i=0$ giving $0=\frac{V}{R}+ce^{0}$

So $c=-\frac{V}{R}$

$i=\frac{V}{R}-\frac{V}{R}e^{-\frac{R}{L}t}=\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right)$

(ii) As $t\to\infty$, $i\to\frac{V}{R}$



(iii) 95% of limiting value is $0.95\frac{V}{R}$

$0.95\frac{V}{R}=\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right)$

So $0.95=1-e^{-\frac{R}{L}t}$

$e^{-\frac{R}{L}t}=0.05$

$t=-\frac{L}{R}\ln 0.05=\frac{L}{R}\ln 20$

which is independent of V.

Edexcel FM First order DEs 2 Exercise

4\. (i) Rewrite the equation

$$\frac{dx}{dt}+\frac{9}{200-t}x=\frac{2}{5}$$

$$\int P(t)\,dt=\int\frac{9}{200-t}dt=-9\ln(200-t)$$

So the integrating factor is $e^{\int P(t)\,dt}=e^{-9\ln(200-t)}=(200-t)^{-9}$

Multiplying by the integrating factor, the equation becomes

$$\frac{d}{dt}\left((200-t)^{-9}x\right)=\frac{2}{5}(200-t)^{-9}$$

$$(200-t)^{-9}x=\int\left(\frac{2}{5}(200-t)^{-9}\right)dt=\frac{2}{5}\frac{(200-t)^{-8}}{(-1)(-8)}+c$$

Rearranging

$$x=\frac{1}{20}(200-t)+c(200-t)^{9}$$

(ii) When $t=0$, $x=5$

$$5=\frac{1}{20}(200)+c(200)^{9}\text{ giving } c=-\frac{5}{200^{9}}$$

So $x=\dfrac{1}{20}(200-t)-\dfrac{5}{200^{9}}(200-t)^{9}$

(iii) The mixture eventually almost reaches the concentration of the mixture being poured in and so the rate at which the salt decreases is proportional to the net rate of decrease of volume which is constant.