

Section 2: Integrating factors

Solutions to Exercise level 1

$$1. \quad (i) \quad x^2 \frac{dy}{dx} + 2xy = \cos x$$

$$\frac{dy}{dx}(x^2 y) = \cos x$$

$$x^2 y = \sin x + c$$

$$y = \frac{1}{x^2} \sin x + \frac{c}{x^2}$$

$$(ii) \quad x^3 e^y \frac{dy}{dx} + 3x^2 e^y = 4$$

$$\frac{d}{dx}(x^3 e^y) = 4$$

$$x^3 e^y = 4x + c$$

$$e^y = \frac{4}{x^2} + \frac{c}{x^3}$$

$$y = \ln \left| \frac{4}{x^2} + \frac{c}{x^3} \right|$$

$$(iii) \quad \frac{2y}{x} \frac{dy}{dx} - \frac{y^2}{x^2} = e^x$$

$$\frac{d}{dx} \left(\frac{y^2}{x} \right) = e^x$$

$$\frac{y^2}{x} = e^x + c$$

$$y = \sqrt{xe^x + cx}$$

$$(iv) \quad \sin x \frac{dy}{dx} + y \cos x = x^2$$

$$\frac{dy}{dx}(y \sin x) = x^2$$

$$y \sin x = \frac{1}{3} x^3 + c$$

$$y = \frac{x^3 + k}{3 \sin x}$$

Edexcel FM First order DEs 2 Exercise

2. (i) $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$

Integrating factor = $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$x \frac{dy}{dx} + y = 1$$

$$\frac{d}{dx}(xy) = 1$$

$$xy = x + c$$

$$y = 1 + \frac{c}{x}$$

(ii) $\frac{dy}{dx} + y \cos x = e^{-\sin x}$

Integrating factor = $e^{\int \cos x dx} = e^{\sin x}$

$$e^{\sin x} \frac{dy}{dx} + ye^{\sin x} \cos x = 1$$

$$\frac{d}{dx}(ye^{\sin x}) = 1$$

$$ye^{\sin x} = x + c$$

$$y = (x + c)e^{-\sin x}$$

(iii) $\frac{dy}{dx} + \frac{y}{x+2} = x-3$

Integrating factor = $e^{\int \frac{1}{x+2} dx} = e^{\ln(x+2)} = x+2$

$$(x+2) \frac{dy}{dx} + y = (x-3)(x+2)$$

$$\frac{d}{dx}(y(x+2)) = x^2 - x - 6$$

$$y(x+2) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + c$$

$$y = \frac{2x^3 - 3x^2 - 36x + k}{6(x+2)}$$

(iv) $x \frac{dy}{dx} - y = x^3 e^{2x}$

$$\frac{dy}{dx} - \frac{y}{x} = x^2 e^{2x}$$

Integrating factor = $e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

Edexcel FM First order DEs 2 Exercise

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = xe^{2x}$$

$$\frac{d}{dx} \left(\frac{y}{x} \right) = xe^{2x}$$

$$\frac{y}{x} = \frac{1}{2} xe^{2x} - \int \frac{1}{2} e^{2x} dx + c$$

$$\frac{y}{x} = \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + c$$

$$y = \frac{1}{2} x^2 e^{2x} - \frac{1}{4} x e^{2x} + cx$$

$$(v) \quad \frac{dy}{dx} + y \cot x = x \operatorname{cosec} x$$

$$\text{Integrating factor} = e^{\int \cot x dx} = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln(\sin x)} = \sin x$$

$$\sin x \frac{dy}{dx} + y \cos x = x$$

$$\frac{d}{dx} (y \sin x) = x$$

$$y \sin x = \frac{1}{2} x^2 + c$$

$$y = \frac{x^2 + k}{2 \sin x}$$

$$(vi) \quad x \frac{dy}{dx} + 2y = x + 1$$

$$\frac{dy}{dx} + \frac{2y}{x} = 1 + \frac{1}{x}$$

$$\text{Integrating factor} = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$x^2 \frac{dy}{dx} + 2xy = x^2 + x$$

$$\frac{d}{dx} (x^2 y) = x^2 + x$$

$$x^2 y = \frac{1}{3} x^3 + \frac{1}{2} x^2 + c$$

$$y = \frac{1}{3} x + \frac{1}{2} + \frac{c}{x^2}$$

Edexcel FM First order DEs 2 Exercise

3. (i) $\frac{dy}{dx} + \frac{y}{x} = e^x$

Integrating factor = $e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$x \frac{dy}{dx} + y = xe^x$$

$$\frac{d}{dx}(xy) = xe^x$$

$$xy = xe^x - \int e^x dx$$

$$xy = xe^x - e^x + A$$

$$y = e^x - \frac{e^x}{x} + \frac{A}{x}$$

$y = 1$ when $x = 1$, so $1 = e - e + A$

$$A = 1$$

Particular solution is $y = e^x - \frac{e^x}{x} + \frac{1}{x}$

using integrating by parts

(ii) $\frac{dy}{dx} + 2xy = e^{-(x-2)^2}$

Integrating factor = $e^{\int 2x dx} = e^{x^2}$

$$e^{x^2} \frac{dy}{dx} + 2xe^{x^2} y = e^{-(x-2)^2} e^{x^2}$$

$$\frac{d}{dx}(e^{x^2} y) = e^{-x^2+4x-4+x^2} = e^{4x-4}$$

$$e^{x^2} y = \frac{1}{4} e^{4x-4} + A$$

$$y = \frac{1}{4} e^{-x^2+4x-4} + Ae^{-x^2}$$

$$y = \frac{1}{4} e^{-(x-2)^2} + Ae^{-x^2}$$

$y = 0$ when $x = 1$, so $0 = \frac{1}{4} e^{-1} + Ae^{-1}$

$$A = -\frac{1}{4}$$

$$y = \frac{1}{4} e^{-(x-2)^2} - \frac{1}{4} e^{-x^2}$$