Section 2: The inverse hyperbolic functions

Solutions to Exercise level 3

1\. (i) $y = \operatorname{arcosh} x$

$\cosh y = x$

$\sinh y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

(ii)
$$\int \frac{1}{\sqrt{x^2-1}} \operatorname{arcosh} x \, dx = \operatorname{arcosh} x \times \operatorname{arcosh} x - \int \frac{1}{\sqrt{x^2-1}} \operatorname{arcosh} x \, dx$$
$$= (\operatorname{arcosh} x)^2 - \int \frac{1}{\sqrt{x^2-1}} \operatorname{arcosh} x \, dx$$
$$2\int \frac{1}{\sqrt{x^2-1}} \operatorname{arcosh} x \, dx = (\operatorname{arcosh} x)^2 + c$$
$$\int \frac{1}{\sqrt{x^2-1}} \operatorname{arcosh} x \, dx = \tfrac{1}{2}(\operatorname{arcosh} x)^2 + c$$

(iii)
$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\int \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx = \operatorname{arsinh} x \times \operatorname{arsinh} x - \int \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx$$
$$= (\operatorname{arsinh} x)^2 - \int \frac{1}{\sqrt{x^2-1}} \operatorname{arsinh} x \, dx$$
$$2\int \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx = (\operatorname{arsinh} x)^2 + c$$
$$\int \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx = \tfrac{1}{2}(\operatorname{arsinh} x)^2 + c$$

2\. (i)
$$\operatorname{arsinh} x - \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2+1}\right) - \ln\left(x + \sqrt{x^2-1}\right)$$
$$= \ln \frac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}}$$

Since $\sqrt{x^2+1} > \sqrt{x^2-1}$ $(x > 1)$, the numerator is greater than the denominator, so $\dfrac{x + \sqrt{x^2+1}}{x + \sqrt{x^2-1}} > 1$ and therefore $\operatorname{arsinh} x - \operatorname{arcosh} x > 0$.

(ii) As $x \to \infty$, $\dfrac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}} \to 1$, so $\operatorname{arsinh} x - \operatorname{arcosh} x \to 0$

(iii) $\operatorname{arsinh} n - \operatorname{arcosh} n < \ln(1.01)$

$$\ln\left(\frac{n+\sqrt{n^2+1}}{n+\sqrt{n^2-1}}\right) < \ln 1.01$$

$$\frac{n+\sqrt{n^2+1}}{n+\sqrt{n^2-1}} < 1.01$$

$$n+\sqrt{n^2+1} < 1.01\left(n+\sqrt{n^2-1}\right)$$

$$n+\sqrt{n^2+1} < 1.01n+1.01\sqrt{n^2-1}$$

$$\sqrt{n^2+1}-1.01\sqrt{n^2-1} < 0.01n$$

Squaring: $n^2+1-2.02\sqrt{(n^4-1)}+1.0201(n^2-1) < 0.0001n^2$

$$2.02n^2-0.0201 < 2.02\sqrt{n^4-1}$$

$$n^2-\frac{0.0201}{2.02} < \sqrt{n^4-1}$$

Squaring again: $n^4-\dfrac{2\times 0.0201}{2.02}n^2+\left(\dfrac{0.0201}{2.02}\right)^2 < n^4-1$

$$1+\left(\frac{0.0201}{2.02}\right)^2 < \frac{2\times 0.0201}{2.02}n^2$$

$n > 7.08$

so the smallest integer is $n = 8$.

3. $y = a\operatorname{arcosh} x + b\operatorname{arsinh} x$

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2-1}}+\frac{b}{\sqrt{1+x^2}}$$

$$y'(2) = \frac{a}{\sqrt{3}}+\frac{b}{\sqrt{5}} = \sqrt{8}-\sqrt{6}$$

$$a\sqrt{5}+b\sqrt{3} = \sqrt{15}\left(\sqrt{8}-\sqrt{6}\right) \qquad (1)$$

$$y'(3) = \frac{a}{\sqrt{8}}+\frac{b}{\sqrt{10}} = \tfrac{1}{2}\left(\sqrt{8}-\sqrt{6}\right)$$

$$a\sqrt{10}+b\sqrt{8} = \tfrac{1}{2}\sqrt{80}\left(\sqrt{8}-\sqrt{6}\right) = 2\sqrt{5}\left(\sqrt{8}-\sqrt{6}\right) \qquad (2)$$

(2) $\quad a\sqrt{10}+b\sqrt{8} = 2\sqrt{5}\left(\sqrt{8}-\sqrt{6}\right)$

(1)$\times\sqrt{2}$ $\quad a\sqrt{10}+b\sqrt{6} = \sqrt{30}\left(\sqrt{8}-\sqrt{6}\right)$

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Subtracting: $b\left(\sqrt{8}-\sqrt{6}\right)=\left(2\sqrt{5}-\sqrt{30}\right)\left(\sqrt{8}-\sqrt{6}\right)$

$$b=2\sqrt{5}-\sqrt{30}$$

Using (2): $a\sqrt{10}+\sqrt{8}\left(2\sqrt{5}-\sqrt{30}\right)=2\sqrt{5}\left(\sqrt{8}-\sqrt{6}\right)$

$$a\sqrt{10}+4\sqrt{10}-2\sqrt{60}=4\sqrt{10}-2\sqrt{30}$$

$$a+4-2\sqrt{6}=4-2\sqrt{3}$$

$$a=2\sqrt{6}-2\sqrt{3}$$