Section 2: The inverse hyperbolic functions

Solutions to Exercise level 2

1. $$4x^2 - 16x + 32 = 4(x^2 - 4x + 8)$$
$$= 4((x-2)^2 - 4 + 8)$$
$$= 4((x-2)^2 + 4)$$
$$\int \frac{dx}{\sqrt{4x^2 - 16x + 32}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x-2)^2 + 4}}$$
$$= \frac{1}{2}\operatorname{arsinh}\frac{x-2}{2} + c$$

2. (i)
$$\cosh x = c$$
$$\tfrac{1}{2}(e^x + e^{-x}) = c$$
$$e^x + e^{-x} = 2c$$
$$e^{2x} - 2ce^x + 1 = 0$$
$$e^x = \frac{2c \pm \sqrt{4c^2 - 4}}{2}$$
$$= c \pm \sqrt{c^2 - 1}$$
$$x = \ln\left(c + \sqrt{c^2 - 1}\right) \text{ or } \ln\left(c - \sqrt{c^2 - 1}\right)$$

$$\left(c - \sqrt{c^2 - 1}\right)\left(c + \sqrt{c^2 - 1}\right) = c^2 - (c^2 - 1) = 1$$
$$\Rightarrow c - \sqrt{c^2 - 1} = \left(c + \sqrt{c^2 - 1}\right)^{-1}$$

$$x = \ln\left(c + \sqrt{c^2 - 1}\right) \text{ or } \ln\left(c + \sqrt{c^2 - 1}\right)^{-1}$$
$$= \pm\ln\left(c + \sqrt{c^2 - 1}\right)$$

(ii)
$$\sinh^2 x + 3\cosh x = 9$$
$$(\cosh^2 x - 1) + 3\cosh x = 9$$
$$\cosh^2 x + 3\cosh x - 10 = 0$$
$$(\cosh x + 5)(\cosh x - 2) = 0$$
$$\cosh x = -5 \text{ or } 2$$

Since $\cosh x \geq 1$, $\cosh x \neq -5$, so $\cosh x = 2$
$$x = \pm\ln\left(2 + \sqrt{2^2 - 1}\right) = \pm\ln\left(2 + \sqrt{3}\right)$$

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3. (i)

$$y = \operatorname{arsinh} x$$
$$\sinh y = x$$
$$\tfrac{1}{2}\left(e^{y} - e^{-y}\right) = x$$
$$e^{y} - e^{-y} = 2x$$
$$e^{2y} - 2xe^{y} - 1 = 0$$
$$e^{y} = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$
$$= x \pm \sqrt{x^2 + 1}$$

Since e^{y} cannot be negative, $e^{y} = x \pm \sqrt{x^2 + 1}$

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$
$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

(ii)

$$\int_0^2 \frac{1}{\sqrt{3x^2 + 4}}\,dx = \frac{1}{\sqrt{3}}\int_0^2 \frac{1}{\sqrt{x^2 + \frac{4}{3}}}\,dx$$
$$= \left[\frac{1}{\sqrt{3}}\ln\left(x + \sqrt{x^2 + \tfrac{4}{3}}\right)\right]_0^2$$
$$= \frac{1}{\sqrt{3}}\left(\ln(2 + \sqrt{4 + \tfrac{4}{3}}) - \ln\sqrt{\tfrac{4}{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\left(\ln(2 + \sqrt{\tfrac{16}{3}}) - \ln\sqrt{\tfrac{4}{3}}\right)$$
$$= \frac{1}{\sqrt{3}}\ln\left(\frac{2 + \frac{4}{\sqrt{3}}}{\frac{2}{\sqrt{3}}}\right)$$
$$= \frac{1}{\sqrt{3}}\ln\left(\sqrt{3} + 2\right)$$

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4. $2x = \sinh u$

$$2\frac{dx}{du} = \cosh u$$

$$\begin{aligned}\int_0^{\sqrt{2}} \sqrt{4x^2+1}\, dx &= \int_{x=0}^{x=\sqrt{2}} \sqrt{\sinh^2 u + 1} \times \tfrac{1}{2}\cosh u\, du \\ &= \int_{x=0}^{x=\sqrt{2}} \tfrac{1}{2}\cosh^2 u\, du \\ &= \int_{x=0}^{x=\sqrt{2}} \tfrac{1}{4}(\cosh 2u + 1)\, du \\ &= \left[\tfrac{1}{8}\sinh 2u + \tfrac{1}{4}u\right]_{x=0}^{x=\sqrt{2}} \\ &= \left[\tfrac{1}{4}\sinh u \cosh u + \tfrac{1}{4}u\right]_{x=0}^{x=\sqrt{2}} \\ &= \left[\tfrac{1}{4}\times 2x\sqrt{1+4x^2} + \tfrac{1}{4}\operatorname{arsinh} 2x\right]_0^{\sqrt{2}} \\ &= \tfrac{1}{2}\sqrt{2}\times 3 + \tfrac{1}{4}\ln(2\sqrt{2}+3) - 0 \\ &= \tfrac{3}{2}\sqrt{2} + \tfrac{1}{4}\ln(2\sqrt{2}+3)\end{aligned}$$

5. (i) $\displaystyle\int \frac{1}{\sqrt{k+x^2}}\,dx$

For $k < 0$, use standard integral leading to an arcosh function

For $k > 0$, use standard integral leading to an arsinh function

For $k = 0$, $\displaystyle\int \frac{1}{\sqrt{0+x^2}}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

(ii) $\displaystyle\int \frac{1}{\sqrt{k-x^2}}\,dx$

For $k \le 0$, function does not exist

For $k > 0$, use standard integral leading to an arcsin function

(iii) $\displaystyle\int \frac{1}{\sqrt{k+2x+x^2}}\,dx = \int \frac{1}{\sqrt{k-1+(x+1)^2}}\,dx$

For $k < 1$, use standard integral leading to an arcosh function

For $k > 1$, use standard integral leading to an arsinh function

For $k = 1$, $\displaystyle\int \frac{1}{\sqrt{0+(x+1)^2}}\,dx = \int \frac{1}{x+1}\,dx = \ln|x+1| + c$

6. (i) $\displaystyle\int \frac{x^{m-1}}{\sqrt{x^{2m}-1}}\,dx$

Let $u = x^m \quad \Rightarrow \dfrac{du}{dx} = mx^{m-1}$

$$\int \frac{mx^{m-1}}{m\sqrt{x^{2m}-1}}\,dx = \int \frac{1}{m\sqrt{u^2-1}}\,du$$
$$= \frac{1}{m}\operatorname{arcosh} u + c$$
$$= \frac{1}{m}\operatorname{arcosh} x^m + c$$

(ii) Let $y = \frac{1}{m}\operatorname{arsinh} x^m + c$

$$\frac{dy}{dx} = \frac{1}{m} \times mx^{m-1} \times \frac{1}{\sqrt{1+x^{2m}}} = \frac{x^{m-1}}{\sqrt{1+x^{2m}}}$$

so $\int \frac{x^{m-1}}{\sqrt{x^{2m}+1}}\,dx = \frac{1}{m}\operatorname{arsinh} x^m + c$

Let $y = \frac{1}{m}\operatorname{artanh} x^m + c$

$$\frac{dy}{dx} = \frac{1}{m} \times mx^{m-1} \times \frac{1}{1-x^{2m}} = \frac{x^{m-1}}{1-x^{2m}}$$

so $\int \frac{x^{m-1}}{1-x^{2m}}\,dx = \frac{1}{m}\operatorname{artanh} x^m + c$

7. (i) $\operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln\left(\frac{1-x+2x}{1-x}\right) = \frac{1}{2}\ln\left(1+\frac{2x}{1-x}\right)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

$$\ln\left(1+\frac{2x}{1-x}\right) = \frac{2x}{1-x} - \frac{1}{2}\left(\frac{2x}{1-x}\right)^2 + \frac{1}{3}\left(\frac{2x}{1-x}\right)^3 - \ldots$$
$$= 2x(1-x)^{-1} - \tfrac{1}{2}\times 4x^2(1-x)^{-2}$$
$$+ \tfrac{1}{3}\times 8x^3(1-x)^{-3} + \ldots$$
$$= 2x(1+x+x^2+\ldots) - 2x^2(1+2x+\ldots)$$
$$+ \tfrac{8}{3}x^3(1+\ldots) + \ldots$$
$$= 2x + 2x^2 + 2x^3 - 2x^2 - 4x^3 + \tfrac{8}{3}x^3 + \ldots$$
$$= 2x + \tfrac{2}{3}x^3 + \ldots$$

$$\operatorname{artanh} x = \frac{1}{2}\ln\left(1+\frac{2x}{1-x}\right) = x + \tfrac{1}{3}x^3 + \ldots$$

(ii) $f(x) = \operatorname{artanh} x$ $\quad f(0) = 0$

$f'(x) = \frac{1}{1-x^2}$ $\quad f'(0) = 1$

$$f''(x) = -\frac{1}{(1-x^2)^2} \times -2x = \frac{2x}{(1-x^2)^2} \qquad f''(0) = 0$$

$$f''(x) = \frac{2(1-x^2)^2 - 2x \times 2(1-x^2) \times -2x}{(1-x^2)^4} \quad f'''(0) = 2$$

Using the general Maclaurin expansion:

$$\begin{aligned} \text{artanh}\, x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= x + \tfrac{2}{6}x^3 + \dots \\ &= x + \tfrac{1}{3}x^3 + \dots \end{aligned}$$