Section 2: The inverse hyperbolic functions

Solutions to Exercise level 1

1. (i) $\operatorname{arsinh} 3 = \ln\left(3+\sqrt{3^2+1}\right) = \ln\left(3+\sqrt{10}\right)$

(ii) $\operatorname{arcosh}\frac{5}{3} = \ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right) = \ln\left(\frac{5}{3}+\sqrt{\frac{16}{9}}\right) = \ln 3$

(iii) $\operatorname{artanh}\frac{5}{8} = \frac{1}{2}\ln\left(\frac{1+\frac{5}{8}}{1-\frac{5}{8}}\right) = \frac{1}{2}\ln\left(\frac{13}{3}\right)$

(iv) $\operatorname{arsinh}(-2) = \ln\left(-2+\sqrt{2^2+1}\right) = \ln\left(-2+\sqrt{5}\right)$

2. (i) $y = \operatorname{arcosh} x$

$\cosh y = x$

$\sinh y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{1-\cosh^2 y}} = \frac{1}{\sqrt{1-x^2}}$

(ii) $\frac{d}{dx}\operatorname{arcosh}(2x+1) = 2\times\frac{1}{\sqrt{(2x+1)^2-1}}$

$= \frac{2}{\sqrt{4x^2+4x+1-1}}$

$= \frac{2}{\sqrt{4x^2+4x}}$

$= \frac{1}{\sqrt{x^2+x}}$

(iii) $\frac{d}{dx}\left(e^{4x}\operatorname{arcosh}(2x+1)\right) = 4e^{4x}\operatorname{arcosh}(2x+1) + \frac{e^{4x}}{\sqrt{x^2+x}}.$

3. (i) $\int \frac{1}{\sqrt{x^2+9}}\,dx = \operatorname{arsinh}\frac{x}{3} + c$

(ii) $\int \frac{1}{\sqrt{x^2-9}}\,dx = \operatorname{arcosh}\frac{x}{3} + c$

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(iii) $\displaystyle\int \frac{1}{\sqrt{9x^2+1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2+\frac{1}{9}}}\,dx$

$$= \frac{1}{3}\operatorname{arsinh}\frac{x}{\frac{1}{3}} + c$$

$$= \frac{1}{3}\operatorname{arsinh}3x + c$$

(iv) $\displaystyle\int \frac{1}{\sqrt{9x^2-1}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{x^2-\frac{1}{9}}}\,dx$

$$= \frac{1}{3}\operatorname{arcosh}\frac{x}{\frac{1}{3}} + c$$

$$= \frac{1}{3}\operatorname{arcosh}3x + c$$

4. (i) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+4}}\,dx = \left[\ln\left(x+\sqrt{x^2+4}\right)\right]_0^1$

$$= \ln(1+\sqrt{5}) - \ln 2$$

$$= 0.481 \text{ (3 s.f.)}$$

(ii) $\displaystyle\int_{0.5}^1 \frac{1}{\sqrt{16x^2-1}}\,dx = \frac{1}{4}\int_{0.5}^1 \frac{1}{\sqrt{x^2-\frac{1}{16}}}\,dx$

$$= \frac{1}{4}\left[\ln\left(x+\sqrt{x^2-\tfrac{1}{16}}\right)\right]_{0.5}^1$$

$$= \frac{1}{4}\left(\ln(1+\tfrac{1}{4}\sqrt{15}) - \ln(\tfrac{1}{2}+\tfrac{1}{4}\sqrt{3}\right)$$

$$= 0.187 \text{ (3 s.f.)}$$

(iii) $\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2+3}}\,dx = \frac{1}{2}\int_1^2 \frac{1}{\sqrt{x^2+\frac{3}{4}}}\,dx$

$$= \frac{1}{2}\left[\ln\left(x+\sqrt{x^2+\tfrac{3}{4}}\right)\right]_1^2$$

$$= \frac{1}{2}\left(\ln(2+\tfrac{1}{2}\sqrt{19}) - \ln(1+\tfrac{1}{2}\sqrt{7})\right)$$

$$= 0.294 \text{ (3 s.f.)}$$