# Section 1: Introducing the hyperbolic functions

## Exercise level 3

1. (i) $\tanh x = \dfrac{\sinh x}{\cosh x}$

$$= \frac{\frac{1}{2}(e^{x} + e^{-x})}{\frac{1}{2}(e^{x} - e^{-x})}$$

$$= \frac{(e^{x} + e^{-x}) \times e^{x}}{(e^{x} - e^{-x}) \times e^{x}}$$

$$= \frac{e^{2x} + 1}{e^{2x} - 1}$$

(ii) $y = a^2 \sinh x + b^2 \cosh x$

$$\frac{dy}{dx} = a^2 \cosh x + b^2 \sinh x$$

At turning point, $a^2 \cosh x + b^2 \sinh x = 0$

$$\tanh x = -\frac{a^2}{b^2} < -1 \text{ since } a^2 > b^2$$

Since tanh x lies between -1 and 1, there are no turning points.

Where the curve crosses the x-axis, $a^2 \sinh x + b^2 \cosh x = 0$

$$\tanh x = -\frac{b^2}{a^2} > -1$$

So the curve does cross the x-axis.

(iii) If $b^2 > a^2$, turning point is given by $\tanh x = -\dfrac{a^2}{b^2} > -1$, so there is a turning point.

x-axis intersection is given by $\tanh x = -\dfrac{b^2}{a^2} < -1$ so there is no intersection.

(iv) If $a^2 = b^2$, any turning points and x-intersections are both given by $\tanh x = -1$, which is not possible, so there are no turning points or x-intersections.

 

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2. (i) $a\cosh x + b\sinh x = c$

$$\frac{a(e^x + e^{-x})}{2} + \frac{b(e^x - e^{-x})}{2} = c$$

$$ae^x + ae^{-x} + be^x - be^{-x} = 2c$$

$$(a+b)e^x + (a-b)e^{-x} - 2c = 0$$

$$(a+b)e^{2x} - 2ce^x + (a-b) = 0$$

$$\text{Determinant} = 4c^2 - 4(a+b)(a-b)$$
$$= 4(c^2 - a^2 + b^2)$$
$$= 4(b^2 + c^2 - a^2)$$

From triangle, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

If A is obtuse, $\cos A < 0$

$$b^2 + c^2 - a^2 < 0$$

so determinant is zero and therefore there are no real roots.

(ii) If A is a right-angle, then $\cos A = 0$

$$b^2 + c^2 - a^2 = 0$$

Determinant is zero so there is one real root

$$(a+b)e^{2x} - 2ce^x + (a-b) = 0$$

From quadratic formula with zero determinant: $e^x = \dfrac{2c}{2(a+b)}$

$$x = \ln\left(\frac{c}{a+b}\right)$$

(iii) If A is acute, $\cos A > 0$

$$b^2 + c^2 - a^2 > 0$$

so determinant is positive and therefore there are two real roots

$$e^x = \frac{2c \pm \sqrt{4(b^2 + c^2 - a^2)}}{2(a+b)}$$
$$= \frac{c \pm \sqrt{b^2 + c^2 - a^2}}{a+b}$$
$$x = \ln\left(\frac{c \pm \sqrt{b^2 + c^2 - a^2}}{a+b}\right)$$

3. (i) $n = 0$ $\quad \dfrac{d}{dx}(1) = 0$

$n = 1$ $\quad \dfrac{d}{dx}(\sinh x \cosh x) = \sinh x \sinh x + \cosh x \cosh x$

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$n=2$ $\qquad \dfrac{d}{dx}(\sinh 2x \cosh^2 x)$

$$= \sinh 2x \times 2\cosh x \sinh x + 2\cosh 2x \cosh^2 x$$

$n=3$ $\qquad \dfrac{d}{dx}(\sinh 3x \cosh^3 x)$

$$= \sinh 3x \times 3\cosh^2 x \sinh x + 3\cosh 3x \cosh^3 x$$

When $x = 0$, $\dfrac{dy}{dx} = 0 + 1 + 2 + 3 = 6$

(ii) $n = 0$ $\qquad \int_0^1 (1+1)\,dx = [2x]_0^1 = 2$

$n = 1$ $\qquad \int_0^1 (\sinh x + \cosh x)\,dx = [\cosh x + \sinh x]_0^1$

$$= \left[e^x\right]_0^1$$

$$= e - 1$$

$n = 2$ $\qquad \int_0^1 (\sinh^2 x + \cosh^2 x)\,dx = \int_0^1 \cosh 2x\,dx$

$$= \left[\tfrac{1}{2}\sinh 2x\right]_0^1$$

$$= \left[\frac{e^{2x} - e^{2x}}{4}\right]_0^1$$

$$= \frac{e^2 - e^{-2}}{4}$$

$n = 3$

$$\int_0^1 (\sinh^3 x + \cosh^3 x)\,dx$$

$$= \int_0^1 (\sinh x(\cosh^2 x - 1) + \cosh x(\sinh^2 x + 1))\,dx$$

$$= \left[\tfrac{1}{3}\cosh^3 x - \cosh x + \tfrac{1}{3}\sinh^3 x + \sinh x\right]_0^1$$

$$= \frac{1}{3}\left(\frac{e+e^{-1}}{2}\right)^3 - \left(\frac{e+e^{-1}}{2}\right) + \frac{1}{3}\left(\frac{e-e^{-1}}{2}\right)^3 + \left(\frac{e-e^{-1}}{2}\right) - \frac{1}{3} + 1$$

$$= \frac{1}{24}\left(e^3 + 3e + 3e^{-1} + e^{-3}\right) + \frac{1}{24}\left(e^3 - 3e + 3e^{-1} - e^{-3}\right) - e^{-1} + \frac{2}{3}$$

$$= \frac{1}{24}\left(2e^3 + 6e^{-1}\right) - e^{-1} + \frac{2}{3}$$

$$= \tfrac{2}{3} + \tfrac{1}{12}e^3 - \tfrac{3}{4}e^{-1}$$

$$\text{Total} = 2 + e - 1 + \tfrac{1}{4}e^2 - \tfrac{1}{4}e^{-2} + \tfrac{2}{3} + \tfrac{1}{12}e^3 - \tfrac{3}{4}e^{-1}$$

$$= \tfrac{5}{3} + e - \tfrac{3}{4}e^{-1} + \tfrac{1}{4}e^2 - \tfrac{1}{4}e^{-2} + \tfrac{1}{12}e^3$$