Section 1: Introducing the hyperbolic functions

Exercise level 2 solutions

1. (i) $2\cosh^2 x - 1 = 2\left\{\frac{1}{2}\left(e^x + e^{-x}\right)\right\}^2 - 1$

$= 2 \times \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - 1$

$= \frac{1}{2}e^{2x}1 + \frac{1}{2}e^{-2x} - 1$

$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$

$= \cosh 2x$

(ii) $y = 7\sinh x - \sinh 2x$

$\frac{dy}{dx} = 7\cosh x - 2\cosh 2x$

$= 7\cosh x - 2(2\cosh^2 x - 1)$

At stationary points, $7\cosh x - 2(2\cosh^2 x - 1) = 0$

$4\cosh^2 x - 7\cosh x - 2 = 0$

$(4\cosh x + 1)(\cosh x - 2) = 0$

$\cosh x = -\frac{1}{4}$ or 2

Since $\cosh x$ cannot be negative, $\cosh x = 2$.

$\cosh^2 x - \sinh^2 x = 1$

$2^2 - \sinh^2 x = 1$

$\sinh^2 x = 3$

$\sinh x = \pm\sqrt{3}$

$y = 7\sinh x - \sinh 2x = 7\sinh x - 2\sinh x\cosh x$

If $\sinh x = \sqrt{3}$, $y = 7\sinh x - 2\sinh x\cosh x$

$= 7\sqrt{3} - 2\sqrt{3} \times 2$

$= 3\sqrt{3}$

If $\sinh x = -\sqrt{3}$, $y = 7\sinh x - 2\sinh x\cosh x$

$= -7\sqrt{3} + 2\sqrt{3} \times 2$

$= -3\sqrt{3}$

(iii) $\int_0^{\ln 3} (7\sinh x - \sinh 2x)dx = \left[7\cosh x - \frac{1}{2}\cosh 2x\right]_0^{\ln 3}$

$= \left[\frac{7}{2}\left(e^x + e^{-x}\right) - \frac{1}{4}\left(e^{2x} + e^{-2x}\right)\right]_0^{\ln 3}$

$= \frac{7}{2}\left(e^{\ln 3} + e^{-\ln 3}\right) - \frac{1}{4}\left(e^{2\ln 3} + e^{-2\ln 3}\right) - 7 + \frac{1}{2}$

$= \frac{7}{2}\left(3 + \frac{1}{3}\right) - \frac{1}{4}\left(9 + \frac{1}{9}\right) - 7 + \frac{1}{2}$

$= \frac{35}{3} - \frac{41}{18} - 7 + \frac{1}{2}$

$= \frac{26}{9}$

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2. (i) $\int_0^{\ln a}(12\cosh x - 8\sinh x)\,dx = \left[12\sinh x - 8\cosh x\right]_0^{\ln a}$

$= \left[12\times\frac{1}{2}(e^x - e^{-x}) - 8\times\frac{1}{2}(e^x + e^{-x})\right]_0^{\ln a}$

$= \left[2e^x - 10e^{-x}\right]_0^{\ln a}$

$= 2e^{\ln a} - 10e^{-\ln a} - (2 - 10)$

$= 2a - \frac{10}{a} + 8$

(ii) $12\cosh x - 8\sinh x = 9$

$12\times\frac{1}{2}(e^x + e^{-x}) - 8\times\frac{1}{2}(e^x - e^{-x}) = 9$

$2e^x + 10e^{-x} = 9$

$2e^{2x} - 9e^x + 10 = 0$

$(2e^x - 5)(e^x - 2) = 0$

$e^x = \frac{5}{2}$ or 2

$x = \ln 2.5$ or $\ln 2$

(iii) $y = 12\cosh x - 8\sinh x$

$\frac{dy}{dx} = 12\sinh x - 8\cosh x$

At stationary points, $12\sinh x - 8\cosh x = 0$

$12\times\frac{1}{2}(e^x - e^{-x}) - 8\times\frac{1}{2}(e^x + e^{-x})$

$2e^x = 10e^{-x}$

$e^{2x} = 5$

$e^x = \sqrt{5}$

$y = 2e^x + 10e^{-x} = 2\sqrt{5} + \frac{10}{\sqrt{5}} = 2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$

$\frac{d^2y}{dx^2} = 12\cosh x - 8\sinh x = 2e^x + 10e^{-x} > 0$, so the stationary point is a minimum point.

Since y is continuous and there are no other turning points, $y \geq 4\sqrt{5}$.

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3. $\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right) = \left(\frac{e^{\frac{A+B}{2}} - e^{-\frac{A+B}{2}}}{2}\right)\left(\frac{e^{\frac{A-B}{2}} - e^{-\frac{A-B}{2}}}{2}\right)$

$= \left(\frac{e^{\frac{A+B}{2}}e^{\frac{A-B}{2}} - e^{-\frac{A+B}{2}}e^{\frac{A-B}{2}} - e^{\frac{A+B}{2}}e^{-\frac{A-B}{2}} + e^{-\frac{A+B}{2}}e^{-\frac{A-B}{2}}}{4}\right)$

$= \left(\frac{e^{A} - e^{-B} - e^{B} + e^{-A}}{4}\right)$

$= \frac{1}{2}\left(\frac{e^{A} + e^{-A}}{2} - \frac{e^{B} + e^{-B}}{2}\right)$

$= \frac{1}{2}\cosh A - \frac{1}{2}\cosh B$

Let $A = 4$ and $B = 2$, so $\frac{A+B}{2} = 3$ and $\frac{A-B}{2} = 1$

$\cosh A - \cosh B = 2\sinh\left(\frac{A+B}{2}\right)\sinh\left(\frac{A-B}{2}\right)$

$\cosh 4 - \cosh 2 = 2\sinh 3 \sinh 1$

4. (i) (a) $y = e^{x}\sinh x$

Using the product rule: $\frac{dy}{dx} = e^{x}\sinh x + e^{x}\cosh x$

(b) $y = e^{x}\sinh x = e^{x}\left(\frac{e^{x} - e^{-x}}{2}\right) = \frac{e^{2x} - 1}{2} = \frac{1}{2}e^{2x} - \frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2} \times 2e^{2x} = e^{2x}$

(ii) $e^{x}\sinh x + e^{x}\cosh x = e^{x}\left(\frac{e^{x} - e^{-x}}{2}\right) + e^{x}\left(\frac{e^{x} + e^{-x}}{2}\right)e^{x}$

$= e^{x}\left(\frac{e^{x} + e^{x}}{2}\right)$

$= e^{2x}$

so they are both the same result.