Section 2: The area of a sector

Exercise level 3 solutions

1. (a) $r = \dfrac{1}{\sin\theta - \cos\theta}$

$r\sin\theta - r\cos\theta = 1$

$y - x = 1$

$r = \dfrac{1}{a\sin\theta - b\cos\theta}$

$ar\sin\theta - br\cos\theta = 1$

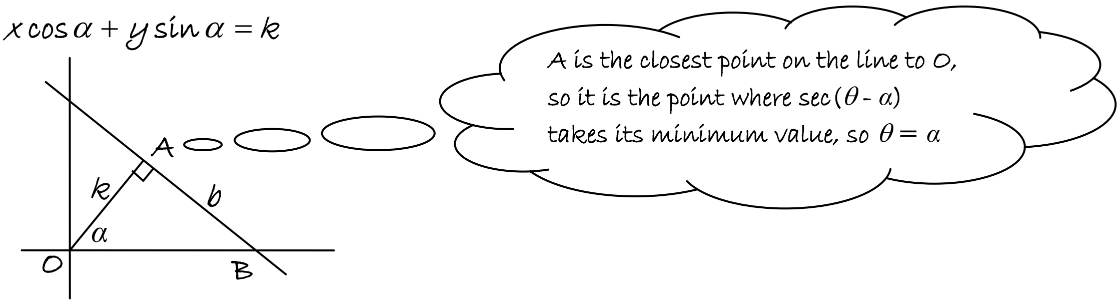
$ay - bx = 1$

(b) $r = k\sec(\theta - \alpha)$

$r\cos(\theta - \alpha) = k$

$r(\cos\theta\cos\alpha + \sin\theta\sin\alpha) = k$

$x\cos\alpha + y\sin\alpha = k$

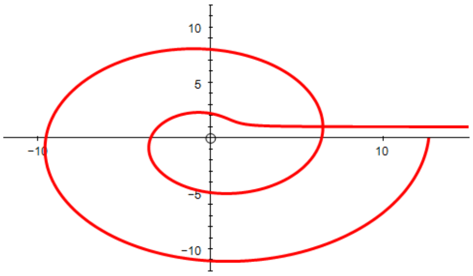


(i) Using basic trigonometry gives $b = k\tan\alpha$

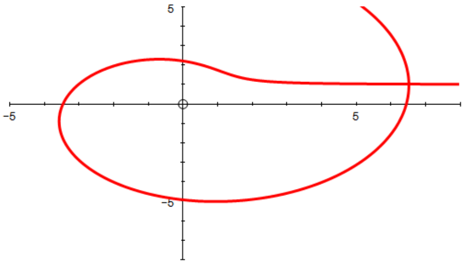
$$\text{Area} = \tfrac{1}{2}kb = \tfrac{1}{2}k \times k\tan\alpha = \tfrac{1}{2}k^2\tan\alpha$$

(ii)
$$\begin{aligned}\text{Area} &= \int_0^\alpha \tfrac{1}{2}r^2\,d\theta \\ &= \int_0^\alpha \tfrac{1}{2}k^2\sec^2(\theta-\alpha)\,d\theta \\ &= \tfrac{1}{2}k^2\left[\tan(\theta-\alpha)\right]_0^\alpha \\ &= 0 - \tfrac{1}{2}k^2\tan(-\alpha) \\ &= \tfrac{1}{2}k^2\tan\alpha\end{aligned}$$

2.



The central loop looks like this:



At the intersection point, $\alpha + \frac{1}{\alpha} = 2\pi + \alpha + \frac{1}{2\pi + \alpha}$

$$\Rightarrow \frac{1}{\alpha} = 2\pi + \frac{1}{2\pi + \alpha}$$
$$\Rightarrow 2\pi + \alpha = 2\pi\alpha(2\pi + \alpha) + \alpha$$
$$\Rightarrow 1 = \alpha(2\pi + \alpha)$$
$$\Rightarrow \alpha^2 + 2\pi\alpha - 1 = 0$$
$$\Rightarrow \alpha = \frac{-2\pi + \sqrt{4\pi^2 + 4}}{2} = -\pi + \sqrt{\pi^2 + 1}$$

Let $\beta = \sqrt{\pi^2 + 1}$

So limits of integration are $-\pi + \beta$ and $2\pi - \pi + \beta = \pi + \beta$

$$\text{Area } = \int_{\beta-\pi}^{\beta+\pi} \tfrac{1}{2}\left(\theta + \frac{1}{\theta}\right)^2 d\theta$$
$$= \tfrac{1}{2}\int_{\beta-\pi}^{\beta+\pi}\left(\theta^2 + 2 + \frac{1}{\theta^2}\right)d\theta$$
$$= \tfrac{1}{2}\left[\tfrac{1}{3}\theta^3 + 2\theta - \frac{1}{\theta}\right]_{\beta-\pi}^{\beta+\pi}$$
$$= \tfrac{1}{6}\left[(\beta+\pi)^3 - (\beta-\pi)^3\right] + (\beta+\pi) - (\beta-\pi) - \tfrac{1}{2}\left[\frac{1}{\beta+\pi} - \frac{1}{\beta-\pi}\right]$$
$$= \tfrac{1}{6}(\beta^3 + 3\beta^2\pi + 3\beta\pi^2 + \pi^3 - \beta^3 + 3\beta^2\pi - 3\beta\pi^2 + \pi^3)$$
$$+ 2\pi - \frac{\beta - \pi - (\beta + \pi)}{2(\beta+\pi)(\beta-\pi)}$$

$$= \beta^2\pi + \tfrac{1}{3}\pi^3 + 2\pi + \frac{\pi}{\beta^2 - \pi^2}$$

$$= (\pi^2 + 1)\pi + \tfrac{1}{3}\pi^3 + 2\pi + \frac{\pi}{\pi^2 + 1 - \pi^2}$$

$$= (\pi^2 + 1)\pi + \tfrac{1}{3}\pi^3 + 2\pi + \pi$$

$$= \tfrac{4}{3}\pi^3 + 4\pi$$