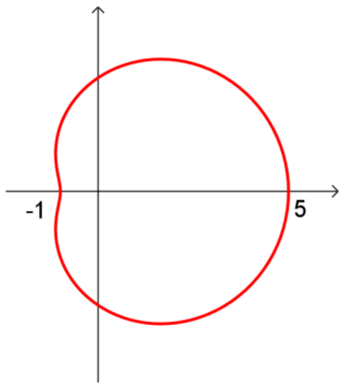
Section 2: The area of a sector

Exercise level 2 solutions

1. (i)



(ii) Area $= \frac{1}{2}\int_0^{2\pi}(3+2\cos\theta)^2 d\theta$

$= \frac{1}{2}\int_0^{2\pi}(9+12\cos\theta+4\cos^2\theta)d\theta$

$= \frac{1}{2}\int_0^{2\pi}(9+12\cos\theta+2(\cos 2\theta+1))d\theta$

$= \frac{1}{2}\int_0^{2\pi}(11+12\cos\theta+2\cos 2\theta)d\theta$

$= \frac{1}{2}[11\theta+12\sin\theta+\sin 2\theta]_0^{2\pi}$

$= 11\pi$

(iii) $x = r\cos\theta = 3\cos\theta + 2\cos^2\theta$

$y = r\sin\theta = 3\sin\theta + 2\cos\theta\sin\theta$

For tangents parallel to the initial line, $\frac{dy}{d\theta} = 0$

$\frac{dy}{d\theta} = 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

$3\cos\theta + 2\cos^2\theta - 2\sin^2\theta = 0$

$3\cos\theta + 2\cos^2\theta - 2 + 2\cos^2\theta = 0$

$4\cos^2\theta + 3\cos\theta - 2 = 0$

$\cos\theta = \frac{-3\pm\sqrt{41}}{8}$

Using a calculator gives $y = \pm 3.485$

The tangents parallel to the initial line are $y = \pm 3.485$

For tangents perpendicular to the initial line, $\frac{dx}{d\theta} = 0$

$\frac{dx}{d\theta} = -3\sin\theta - 4\cos\theta\sin\theta$

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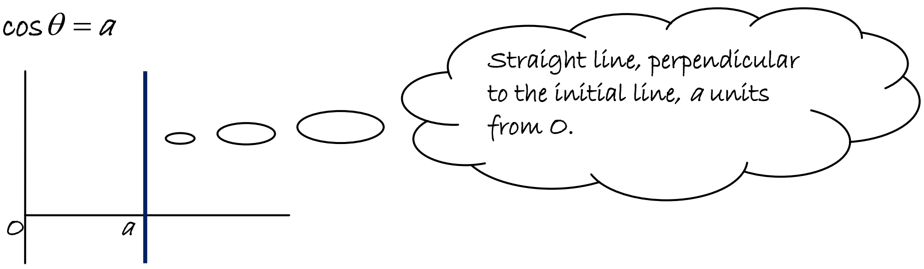
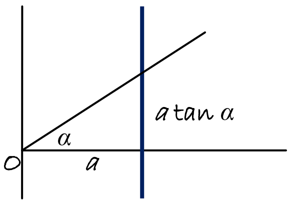
$-3\sin\theta - 4\cos\theta\sin\theta = 0$
$\sin\theta(3 + 4\cos\theta) = 0$
$\sin\theta = 0$ or $\cos\theta = -\frac{3}{4}$
$\sin\theta = 0 \Rightarrow \theta = 0$ or $\theta = \pi$
$\theta = 0 \Rightarrow x = 3\cos 0 + 2\cos^2 0 = 3 + 2 = 5$
$\theta = \pi \Rightarrow x = 3\cos\pi + 2\cos^2\pi = -3 + 2 = -1$
$\cos\theta = -\frac{3}{4} \Rightarrow x = 3 \times -\frac{3}{4} + 2 \times \left(-\frac{3}{4}\right)^2 = -\frac{9}{8}$
The tangents perpendicular to the initial line are $x = 5$, $x = -1$ and $x = -\frac{9}{8}$

2. (i) $r\cos\theta = a$



Cartesian equation is $x = a$

(ii) (a) $A = \frac{1}{2} \times a \times a\tan\alpha$
$= \frac{1}{2}a^2\tan\alpha$



(b) $A = \int_0^\alpha \frac{1}{2}r^2 d\theta$
$= \frac{1}{2}\int_0^\alpha \frac{a^2}{\cos^2\theta} d\theta$
$= \frac{1}{2}a^2\int_0^\alpha \sec^2\theta \, d\theta$
$= \frac{1}{2}a^2\left[\tan\theta\right]_0^\alpha$
$= \frac{1}{2}a^2\tan\alpha$

3. (i) $r = \sin\theta - \cot\theta$
$\frac{dr}{d\theta} = \cos\theta + \operatorname{cosec}^2\theta$
Since $\frac{dr}{d\theta}$ is positive for the values $1 \le \theta \le \frac{\pi}{2}$, it is increasing. Since $r > 0$ for $\theta = 1$, r is positive over this range.

(ii) $A = \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sin\theta - \cot\theta)^2 d\theta$

$= \int_{\pi/3}^{\pi/2} \frac{1}{2}(\sin^2\theta - 2\cos\theta + \cot^2\theta)\,d\theta$

$= \frac{1}{2}\int_{\pi/3}^{\pi/2} (\frac{1}{2}(1-\cos 2\theta) - 2\cos\theta + \operatorname{cosec}^2\theta - 1)\,d\theta$

$= \frac{1}{2}\int_{\pi/3}^{\pi/2} (-\frac{1}{2}\cos 2\theta - 2\cos\theta + \operatorname{cosec}^2\theta - \frac{1}{2})\,d\theta$

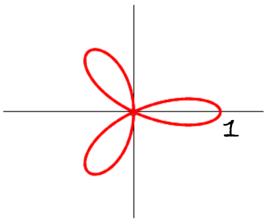
$= \frac{1}{2}\left[-\frac{1}{4}\sin 2\theta - 2\sin\theta - \cot\theta - \frac{1}{2}\theta\right]_{\pi/3}^{\pi/2}$

$= -\frac{1}{8}(\sin\pi - \sin\frac{2\pi}{3}) - (\sin\frac{\pi}{2} - \sin\frac{\pi}{3}) - \frac{1}{2}(\cot\frac{\pi}{2} - \cot\frac{\pi}{3}) - \frac{1}{4}(\frac{\pi}{2} - \frac{\pi}{3})$

$= 0 + \frac{\sqrt{3}}{16} - 1 + \frac{\sqrt{3}}{2} - 0 + \frac{1}{2\sqrt{3}} - \frac{\pi}{24}$

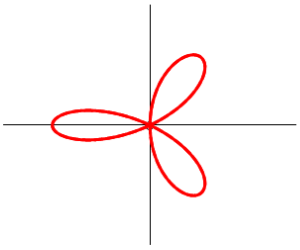
$= \frac{35\sqrt{3}}{48} - 1 - \frac{\pi}{24}$

4. (i)



(ii) The effect of adding a is to rotate the three loops about the origin through an angle of $\frac{a}{3}$ anticlockwise. So the next time the curve will be symmetrical about the initial line, the curve has been rotated through $\frac{\pi}{3}$, so $a = \pi$.

(iii)



(iv) $\cos 3\theta = 0 \Rightarrow \theta = \frac{\pi}{6}$, so using limits of 0 and $\frac{\pi}{6}$ gives the area of half a loop.

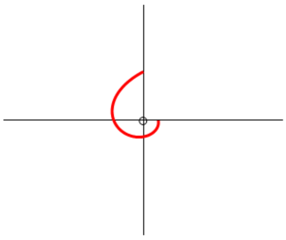
$$\text{Total area } = 6\int_0^{\pi/6} \tfrac{1}{2} r^2 d\theta$$
$$= 3\int_0^{\pi/6} \cos^2 3\theta \, d\theta$$
$$= 3\int_0^{\pi/6} \tfrac{1}{2}(\cos 6\theta + 1) d\theta$$
$$= \tfrac{3}{2}\left[\tfrac{1}{2}\sin 6\theta + \theta\right]_0^{\pi/6}$$
$$= \tfrac{3}{2}\left(0 + \tfrac{\pi}{6} - 0\right) = \frac{\pi}{4}$$

5. (i) $u = \cos x \Rightarrow \dfrac{du}{dx} = -\sin x$

$$\int \operatorname{cosec} x \, dx = \int \operatorname{cosec} x \times \frac{-1}{\sin x} du = \int -\frac{1}{\sin^2 x} du$$
$$= \int -\frac{1}{1-u^2} du$$
$$= \int -\frac{1}{(1+u)(1-u)} du$$
$$= \int \frac{1}{2(1+u)} - \frac{1}{2(1-u)} du$$
$$= \tfrac{1}{2}\ln|1+u| - \tfrac{1}{2}\ln|1-u| + c$$
$$= \tfrac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right| + c$$

(ii) $$\text{Area } = \tfrac{1}{2}\int_{\pi/3}^{\pi/2}\left(1 + \frac{1}{\sin\theta}\right)^2 d\theta$$
$$= \tfrac{1}{2}\int_{\pi/3}^{\pi/2}\left(1 + 2\operatorname{cosec}\theta + \operatorname{cosec}^2\theta\right) d\theta$$
$$= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\ln\left|\frac{1+\cos\theta}{1-\cos\theta}\right| - \cot\theta\right]_{\pi/3}^{\pi/2}$$
$$= \tfrac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + \tfrac{1}{4}\ln\frac{1}{1} - \tfrac{1}{4}\ln\frac{1.5}{0.5} - \tfrac{1}{2}\left(0 - \frac{1}{\sqrt{3}}\right)$$
$$= \tfrac{\pi}{12} - \tfrac{1}{4}\ln 3 + \tfrac{1}{6}\sqrt{3}$$

6.



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$$\begin{aligned}
Area &= \tfrac{1}{2}\int_{\pi/2}^{2\pi} \frac{a^2}{\theta^2}\,d\theta \\
&= \tfrac{1}{2}a^2\left[-\frac{1}{\theta}\right]_{\pi/2}^{2\pi} \\
&= \tfrac{1}{2}a^2\left(-\frac{1}{2\pi}+\frac{2}{\pi}\right) \\
&= \tfrac{1}{2}a^2 \times \frac{3}{2\pi}
\end{aligned}$$

$$Area = 1 \text{ so } \frac{3}{4\pi}a^2 = 1 \Rightarrow a = 2\sqrt{\frac{\pi}{3}}$$