Section 2: The area of a sector

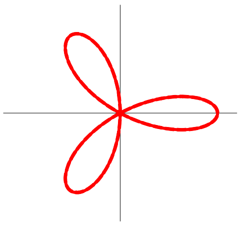
Exercise level 1 solutions

1. (i) $r = a\cos 3\theta$

When $r = 0$, $\cos 3\theta = 0$

$3\theta = \pm\frac{\pi}{2}$

$\theta = \pm\frac{\pi}{6}$



One loop of the curve lies between $\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$.

By symmetry, area of one loop $= 2\int_0^{\frac{\pi}{6}} \frac{1}{2} r^2 \, d\theta$

$= \int_0^{\frac{\pi}{6}} a^2 \cos^2 3\theta \, d\theta$

$= a^2 \int_0^{\frac{\pi}{6}} \frac{1}{2}(\cos 6\theta + 1) \, d\theta$

$= \frac{1}{2} a^2 \left[\frac{1}{6}\sin 6\theta + \theta\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2} a^2 \left(\frac{1}{6}\sin \pi + \frac{\pi}{6} - 0\right)$

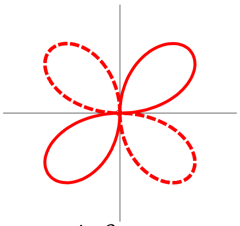
$= \frac{1}{12}\pi a^2$

(ii) $r = a\sin 2\theta$

When $r = 0$, $\sin 2\theta = 0$

$2\theta = 0, \pi$

$\theta = 0, \frac{\pi}{2}$



One loop of the curve lies between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

Area of one loop $= \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2 \sin^2 2\theta \, d\theta$

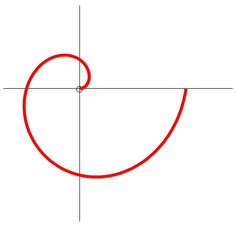
$= \frac{1}{2} a^2 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos 4\theta) \, d\theta$

$= \frac{1}{4} a^2 \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{4} a^2 \left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi - 0\right)$

$= \frac{1}{8}\pi a^2$

2. Area $= \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$



$= \int_0^{\pi} \frac{1}{2}(ae^{\theta})^2 \, d\theta$

$= \frac{1}{2} a^2 \int_0^{\pi} e^{2\theta} \, d\theta$

$= \frac{1}{2} a^2 \left[\frac{1}{2} e^{2\theta}\right]_0^{\pi}$

$= \frac{1}{4} a^2 (e^{2\pi} - 1)$

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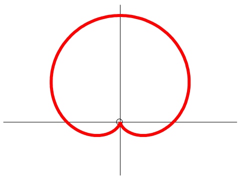
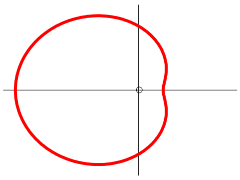
3. (i) $r = 1 + \sin\theta$



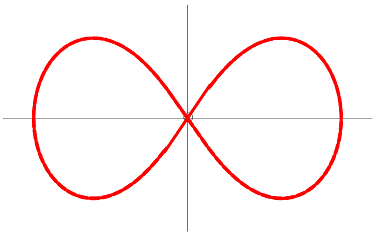
$$\begin{aligned}\text{Area} &= \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2}(1+\sin\theta)^2 \, d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta) \, d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)) \, d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (\tfrac{3}{2} + 2\sin\theta - \tfrac{1}{2}\cos 2\theta) \, d\theta \\ &= \tfrac{1}{2}\left[\tfrac{3}{2}\theta - 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= \tfrac{1}{2}\left[\tfrac{3}{2}\times 2\pi - 2 - 0 - (0 - 2 - 0)\right] \\ &= \tfrac{3}{2}\pi\end{aligned}$$

(ii) $r = 3 - 2\cos\theta$



$$\begin{aligned}\text{Area} &= \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta \\ &= \int_0^{2\pi} \tfrac{1}{2}(3 - 2\cos\theta)^2 \, d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (9 - 12\cos\theta + 4\cos^2\theta) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (9 - 12\cos\theta + 2(1+\cos 2\theta)) d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} (11 - 12\cos\theta + 2\cos 2\theta) d\theta \\ &= \tfrac{1}{2}\left[11\theta - 12\sin\theta + \sin 2\theta\right]_0^{2\pi} \\ &= \tfrac{1}{2}(22\pi - 0) \\ &= 11\pi\end{aligned}$$

(iii) $r^2 = a^2 \cos 2\theta$



When $r = 0$, $\cos 2\theta = 0$

$$2\theta = \pm\tfrac{\pi}{2}$$

$$\theta = \pm\tfrac{\pi}{4}$$

One loop of the curve lies between $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.

By symmetry, area $= 4\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 \, d\theta$

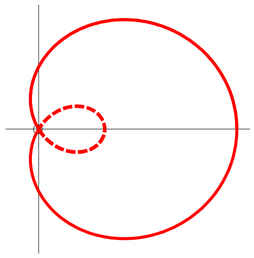
$$= 2\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta$$

$$= 2a^2 \left[\tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= a^2 (\sin \tfrac{\pi}{2} - 0)$$

$$= a^2$$

4. (i) $r = 1 + 2\cos\theta$



(ii) At the ends of the inner loop, $r = 0$, so $1 + 2\cos\theta = 0$

$$\cos\theta = -\tfrac{1}{2}$$

$$\theta = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}$$

Area of inner loop

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \tfrac{1}{2} r^2 \, d\theta$$

$$= \tfrac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 2\cos\theta)^2 \, d\theta$$

$$= \tfrac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 4\cos\theta + 4\cos^2\theta) \, d\theta$$

$$= \tfrac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (1 + 4\cos\theta + 2(1 + \cos 2\theta)) \, d\theta$$

$$= \tfrac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3 + 4\cos\theta + 2\cos 2\theta) \, d\theta$$

$$= \tfrac{1}{2}\left[3\theta + 4\sin\theta + \sin 2\theta\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}$$

$$= \tfrac{1}{2}\left[3 \times \tfrac{4\pi}{3} + 4\sin\tfrac{4\pi}{3} + \sin\tfrac{8\pi}{3} - \left(3 \times \tfrac{2\pi}{3} + 4\sin\tfrac{2\pi}{3} + \sin\tfrac{4\pi}{3}\right)\right]$$

$$= \tfrac{1}{2}\left(4\pi + 4 \times -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3} - 2\pi - 4 \times \tfrac{1}{2}\sqrt{3} - (-\tfrac{1}{2}\sqrt{3})\right)$$

$$= \tfrac{1}{2}(2\pi - 3\sqrt{3})$$

$$= \pi - \tfrac{3}{2}\sqrt{3}$$

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5. $\text{Area} = \int_{\pi}^{2\pi} \tfrac{1}{2} r^2 \, d\theta$

$$= \tfrac{1}{2}\int_{\pi}^{2\pi} \left(\sqrt{\theta} + \frac{1}{\sqrt{\theta}}\right)^2 d\theta$$

$$= \tfrac{1}{2}\int_{\pi}^{2\pi} \left(\theta + 2 + \frac{1}{\theta}\right) d\theta$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\theta^2 + 2\theta + \ln\theta\right]_{\pi}^{2\pi}$$

$$= \tfrac{1}{4}(4\pi^2 - \pi^2) + (2\pi - \pi) + \tfrac{1}{2}(\ln 2\pi - \ln\pi)$$

$$= \tfrac{3}{4}\pi^2 + \pi + \tfrac{1}{2}\ln 2$$