Section 1: Polar coordinates and curves

Exercise level 3 solutions

1. (i) $r = e(a + r\sin\theta)$

$r = ae + er\sin\theta$

$r(1 - e\sin\theta) = ae$

$r = \dfrac{ae}{1 - e\sin\theta}$

(ii) $\sqrt{x^2 + y^2} = e(a + y)$

$x^2 + y^2 = e^2(a + y)^2$

(iii) If $e = 1$, $x^2 + y^2 = (a + y)^2 = a^2 + 2ay + y^2$

$x^2 = a^2 + 2ay$

$y = \dfrac{x^2}{2a} - \dfrac{a}{2}$

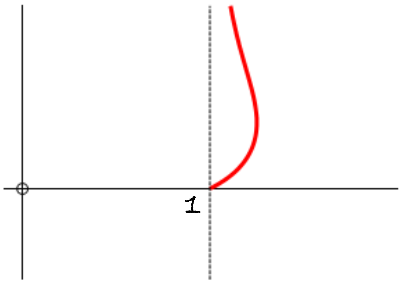
so the curve is a parabola.

Polar equation is $r = \dfrac{a}{1 - \sin\theta}$

(iv) As e varies, obtain different conics – circle, ellipse, parabola, hyperbola and a pair of straight lines

2.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	1	$\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} + 1$	$\frac{1}{2} + \sqrt{3}$	∞



$r = \cos\theta + \tan\theta$

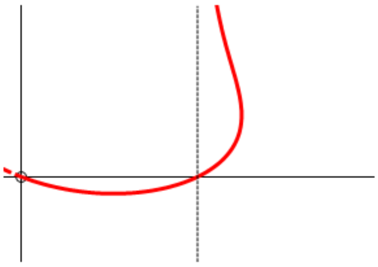
$r\cos\theta = \cos^2\theta + \sin\theta$

$= 1 - \sin^2\theta + \sin\theta$

Since $\sin\theta > \sin^2\theta$, $r\cos\theta > 1$

$\dfrac{dr}{d\theta} = -\sin\theta + \sec^2\theta$

Since $\sec^2\theta > 1$ and $\sin\theta < 1$ over this range, $\dfrac{dr}{d\theta} > 0$ so r is increasing as θ increases.



When $r = 0$, $\cos\theta + \dfrac{\sin\theta}{\cos\theta} = 0$

$$\cos^2\theta + \sin\theta = 0$$

$$1 - \sin^2\theta + \sin\theta = 0$$

$$\sin^2\theta - \sin\theta - 1 = 0$$

$$\sin\theta = \frac{1 \pm \sqrt{5}}{2}$$

$$\theta = \arcsin\left(\frac{1-\sqrt{5}}{2}\right) = -0.666 \text{ rad}$$

3. (i) $r^2 = 2\operatorname{cosec} 2\theta = \dfrac{2}{\sin 2\theta} = \dfrac{2}{2\sin\theta\cos\theta}$

$$r^2 \sin\theta\cos\theta = 1$$

$$r\sin\theta \times r\cos\theta = 1$$

$$yx = 1$$

$$y = \frac{1}{x}$$

(ii) $y = \dfrac{1}{x+y}$

$$y(x+y) = 1$$

$$r\sin\theta(r\cos\theta + r\sin\theta) = 1$$

$$r^2\sin\theta(\cos\theta + \sin\theta) = 1$$

$$r^2 = \frac{1}{\sin\theta(\cos\theta + \sin\theta)}$$

(iii) $r^2 = \dfrac{1}{\sin\theta \times \sqrt{2}(\sin(\theta + \frac{\pi}{4}))}$

$$r^2 = \frac{\operatorname{cosec}\theta \operatorname{cosec}(\theta + \frac{\pi}{4})}{\sqrt{2}}$$

(iv) Using the Cartesian equation: $x = \dfrac{1-y^2}{y} = \dfrac{(1-y)(1+y)}{y}$

so (0, 1) and (0, -1) are on the curve
and for large y, $x \approx y$

Using the polar equation, $r = \infty$ at $\theta = 0, -\frac{\pi}{4}, \pi, \frac{3\pi}{4}$
and $r = 1$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$