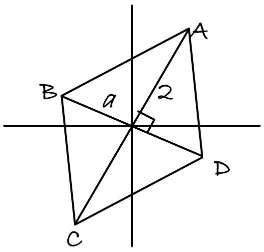
Section 1: Polar coordinates and curves

Exercise level 2 solutions

1. (i) Area of rhombus $= 4 \times \frac{1}{2} \times 2 \times a = 4a$

$4a = 5 \Rightarrow a = 1.25$



For B, $\theta = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$

B = $(1.25, \frac{5\pi}{6})$

For C, $\theta = \frac{5\pi}{6} + \frac{\pi}{2} = \frac{4\pi}{3}$

C = $(2, \frac{4\pi}{3})$

For D, $\theta = \frac{4\pi}{3} + \frac{\pi}{2} = \frac{11\pi}{6}$

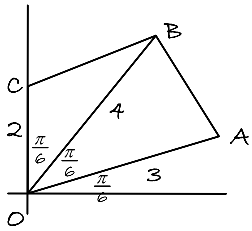
D = $(1.25, \frac{11\pi}{6})$

(ii) Area = area OAB + area OBC

$= \frac{1}{2} \times 3 \times 4 \sin\frac{\pi}{6} + \frac{1}{2} \times 4 \times 2 \sin\frac{\pi}{6}$

$= 3 + 2$

$= 5$

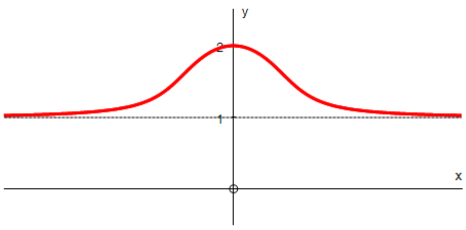


2. (i) $f(-\theta) = \sin(-\theta) + \dfrac{1}{\sin(-\theta)} = -\sin\theta - \dfrac{1}{\sin\theta} = -f(\theta)$

$f(\pi - \theta) = \sin(\pi - \theta) + \dfrac{1}{\sin(\pi - \theta)} = \sin\theta + \dfrac{1}{\sin\theta} = f(\theta)$

(ii) $y = r\sin\theta = \sin^2\theta + 1 > 1$

(iii)



The curve is the same for $\pi < \theta < 2\pi$.

(iv) $\dfrac{dr}{d\theta} = \cos\theta - \dfrac{\cos\theta}{\sin^2\theta} = \dfrac{\cos\theta(\sin^2\theta - 1)}{\sin^2\theta} = -\dfrac{\cos^3\theta}{\sin^2\theta}$

$\dfrac{dr}{d\theta} = 0$ when $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ (or $\frac{3\pi}{2}$ which gives the same point)

So the closest point on the curve to O is $\left(2, \frac{\pi}{2}\right)$.

3. (i) $f(\theta) = \frac{1}{\theta} + \frac{1}{\pi - \theta} \Rightarrow f(\pi - \theta) = \frac{1}{\pi - \theta} + \frac{1}{\theta} = f(\theta)$

so $\theta = \frac{\pi}{2}$ is a line of symmetry.

(ii) $\frac{dr}{d\theta} = -\frac{1}{\theta^2} + \frac{1}{(\pi - \theta)^2}$

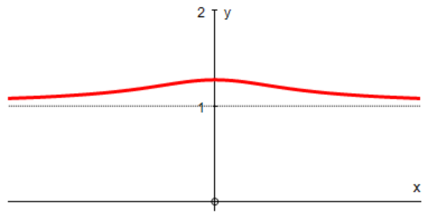
At $\theta = \frac{\pi}{2}$, $\frac{dr}{d\theta} = -\frac{1}{\left(\frac{\pi}{2}\right)^2} + \frac{1}{\left(\frac{\pi}{2}\right)^2} = 0$

For $\theta < \frac{\pi}{2}$, $\frac{dr}{d\theta} < 0$

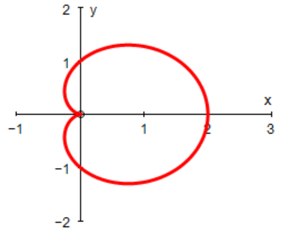
For $\theta > \frac{\pi}{2}$, $\frac{dr}{d\theta} > 0$

so r takes a minimum value at $\theta = \frac{\pi}{2}$.

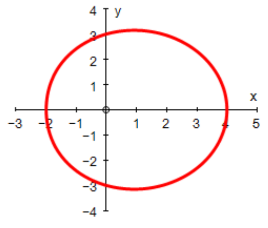
(iii)



4. (i) $r = \cos\theta + 1$



(ii) $r = \cos\theta + 3$



$$r = \cos\theta + k$$

$$r - \frac{r\cos\theta}{r} = k$$

$$\sqrt{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}} = k$$

$$\left(\sqrt{x^2+y^2} - \frac{x}{\sqrt{x^2+y^2}}\right)^2 = k^2$$

$$x^2 + y^2 - 2x + \frac{x^2}{x^2+y^2} = k^2$$

$$(x-1)^2 - 1 + y^2 + \frac{x^2}{x^2+y^2} = k^2$$

When k is large, r is large and so $\frac{x^2}{x^2+y^2}$ is small, and 1 is also small.

so $(x-1)^2 + y^2 \approx k^2$ for large k.

5. (i) $\cos\theta - \sin\theta = 0$

$\Rightarrow \tan\theta = 1$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$

(ii) $r = a(\cos\theta - \sin\theta) = a\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

Maximum value of $|r|$ is $a\sqrt{2}$

This value occurs when $\cos\left(\theta + \frac{\pi}{4}\right) = \pm 1$

$\Rightarrow \theta + \frac{\pi}{4} = 0, \pi, 2\pi$

$\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

(iii) $r = a(\cos\theta - \sin\theta)$

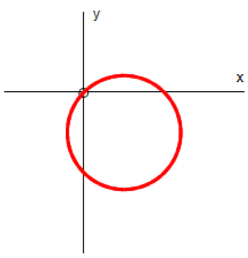
$r^2 = a(r\cos\theta - r\sin\theta)$

$x^2 + y^2 = ax - ay$

$\left(x - \frac{1}{2}a\right)^2 - \frac{1}{4}a^2 + \left(y + \frac{1}{2}a\right)^2 - \frac{1}{4}a^2 = 0$

$\left(x - \frac{1}{2}a\right)^2 + \left(y + \frac{1}{2}a\right)^2 = \frac{1}{2}a^2$

This is a circle.



6. (i) $\dfrac{r^2\cos^2\theta}{4} + r^2\sin^2\theta = 1$

$r^2\cos^2\theta + 4r^2\sin^2\theta = 4$

$r^2 = \dfrac{4}{\cos^2\theta + 4\sin^2\theta}$

(ii) $\left(\dfrac{2}{\sqrt{5}}\sec\theta\right)^2 = \dfrac{4}{\cos^2\theta + 4\sin^2\theta}$

$\dfrac{4}{5\cos^2\theta} = \dfrac{4}{\cos^2\theta + 4\sin^2\theta}$

$\cos^2\theta + 4\sin^2\theta = 5\cos^2\theta$

$4\sin^2\theta = 4\cos^2\theta$

$\sin\theta = \pm\cos\theta$

$\theta = \pm\frac{\pi}{4}$

$r = \dfrac{2}{\sqrt{5}} \times \sqrt{2} = \dfrac{2\sqrt{2}}{\sqrt{5}}$

Intersection points are $\left(\dfrac{2\sqrt{2}}{\sqrt{5}}, \dfrac{\pi}{4}\right)$ and $\left(\dfrac{2\sqrt{2}}{\sqrt{5}}, -\dfrac{\pi}{4}\right)$