Section 1: Further volumes of revolution

Solutions to Exercise

1. $y = e^x$

$$\text{Volume} = \int_0^1 \pi y^2 \, dx$$
$$= \pi \int_0^1 (e^x)^2 \, dx$$
$$= \pi \int_0^1 e^{2x} \, dx$$
$$= \pi \left[\tfrac{1}{2} e^{2x} \right]_0^1$$
$$= \tfrac{1}{2} \pi \left(e^2 - 1 \right)$$

2. $y = \ln(1+x) \quad \Rightarrow e^y = 1 + x \quad \Rightarrow x = e^y - 1$

When $x = 0$, $y = \ln 1 = 0$

When $x = 1$, $y = \ln 2$

$$\text{Volume } V = \int_{\frac{1}{2}}^1 \pi x^2 \, dy$$
$$= \pi \int_0^{\ln 2} \left(e^y - 1 \right)^2 dy$$
$$= \pi \int_0^{\ln 2} \left(e^{2y} - 2e^y + 1 \right) dy$$

$$V = \pi \left[\tfrac{1}{2} e^{2y} - 2e^y + y \right]_0^{\ln 2}$$
$$= \pi \left(\tfrac{1}{2} e^{2\ln 2} - 2e^{\ln 2} + \ln 2 \right) - \pi \left(\tfrac{1}{2} - 2 + 0 \right)$$
$$= \pi \left(\tfrac{1}{2} \times 4 - 2 \times 2 + \ln 2 - \tfrac{1}{2} + 2 \right)$$
$$= \pi (2 - 4 - \tfrac{1}{2} + 2 + \ln 2)$$
$$= \pi (\ln 2 - \tfrac{1}{2})$$

3. $x = t + t^2$, $y = t - t^2$

When the curve crosses the x-axis, $y = 0$, so $t - t^2 = 0$

$$t(1-t) = 0$$
$$t = 0 \text{ or } 1$$

$$x = t + t^2 \quad \Rightarrow \frac{dx}{dt} = 1 + 2t$$

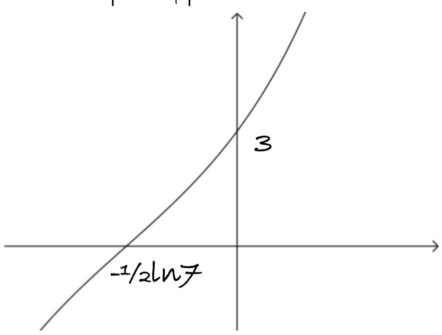
$$\text{Volume} = \int_0^1 \pi y^2 \frac{dx}{dt} \, dt$$

$$= \pi\int_0^1 (t - t^2)^2(1+2t)\,dt$$
$$= \pi\int_0^1 (t^2 - 2t^3 + t^4)(1+2t)\,dt$$
$$= \pi\int_0^1 (t^2 - 2t^3 + t^4 + 2t^3 - 4t^4 + 2t^5)\,dt$$
$$= \pi\int_0^1 (t^2 - 3t^4 + 2t^5)\,dt$$
$$= \pi\left[\tfrac{1}{3}t^3 - \tfrac{3}{5}t^5 + \tfrac{1}{3}t^6\right]_0^1$$
$$= \pi\left(\tfrac{1}{3} - \tfrac{3}{5} + \tfrac{1}{3} - 0\right)$$
$$= \tfrac{1}{15}\pi$$

4. (i) When $x = 0$, $y = 3$

When $y = 0$, $\tanh x = -\frac{3}{4}$

$$x = \tfrac{1}{2}\ln\left|\frac{1-\frac{3}{4}}{1+\frac{3}{4}}\right| = \tfrac{1}{2}\ln\tfrac{1}{7} = -\tfrac{1}{2}\ln 7$$



(ii)
$$(4\sinh x + 3\cosh x)^2 = 16\sinh^2 x + 24\sinh x\cosh x + 9\cosh^2 x$$
$$= 8(\cosh 2x - 1) + 12\sinh 2x + \tfrac{9}{2}(\cosh 2x + 1)$$
$$= \tfrac{25}{2}\cosh 2x + 12\sinh 2x - \tfrac{7}{2}$$

(iii)
$$\text{Volume} = \pi\int_{-\frac{1}{2}\ln 7}^{0} y^2\,dx$$
$$= \pi\int_{-\frac{1}{2}\ln 7}^{0} \left(\tfrac{25}{2}\cosh 2x + 12\sinh 2x - \tfrac{7}{2}\right)dx$$
$$= \pi\left[\tfrac{25}{4}\sinh 2x + 6\cosh 2x - \tfrac{7}{2}x\right]_{-\frac{1}{2}\ln 7}^{0}$$
$$= \pi\left(6 - \tfrac{25}{4}\sinh(-\ln 7) - 6\cosh(-\ln 7) - \tfrac{7}{4}\ln 7\right)$$
$$= \pi\left(6 + \frac{25(7 - \frac{1}{7})}{8} - \frac{6\left(7 + \frac{1}{7}\right)}{2} - \tfrac{7}{4}\ln 7\right)$$
$$= \pi\left(6 - \tfrac{7}{4}\ln 7\right)$$