Section 3: Further integration

Exercise level 2 solutions

1. (i) $$\int \frac{1}{3+4x+4x^2}dx = \frac{1}{4}\int \frac{1}{\frac{3}{4}+x+x^2}dx$$
$$= \frac{1}{4}\int \frac{1}{\frac{3}{4}+(x+\frac{1}{2})^2-\frac{1}{4}}dx$$
$$= \frac{1}{4}\int \frac{1}{(x+\frac{1}{2})^2+\frac{1}{2}}dx$$
$$= \frac{1}{4}\times\frac{1}{1/\sqrt{2}}\arctan\frac{x+\frac{1}{2}}{1/\sqrt{2}}+c$$
$$= \frac{\sqrt{2}}{4}\arctan\frac{\sqrt{2}(2x+1)}{2}+c$$

(ii) $$\int \frac{1}{\sqrt{3-4x-4x^2}}dx = \frac{1}{2}\int \frac{1}{\sqrt{\frac{3}{4}-x-x^2}}dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{3}{4}-(x+\frac{1}{2})^2+\frac{1}{4}}}dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{1-(x+\frac{1}{2})^2}}dx$$
$$= \frac{1}{2}\arcsin(x+\tfrac{1}{2})+c$$

(iii) $$8+6x-9x^2 = 8-9(x^2-\tfrac{2}{3}x)$$
$$= 8-9[(x-\tfrac{1}{3})^2-\tfrac{1}{9}]$$
$$= 8-9(x-\tfrac{1}{3})^2+1$$
$$= 9-9(x-\tfrac{1}{3})^2$$
$$\int \frac{1}{\sqrt{8+6x-9x^2}}dx = \int \frac{1}{\sqrt{9-9(x-\frac{1}{3})^2}}dx$$
$$= \frac{1}{3}\int \frac{1}{\sqrt{1-(x-\frac{1}{3})^2}}dx$$
$$= \tfrac{1}{3}\arcsin(x-\tfrac{1}{3})+c$$

(iv) $$4x^2-4x+5 = 4(x^2-x)+5$$
$$= 4[(x-\tfrac{1}{2})^2-\tfrac{1}{4}]+5$$
$$= 4(x-\tfrac{1}{2})^2-1+5$$
$$= 4(x-\tfrac{1}{2})^2+4$$

$$\int \frac{1}{4x^2 - 4x + 5}\,dx = \int \frac{1}{4(x-\frac{1}{2})^2 + 4}\,dx$$
$$= \frac{1}{4}\int \frac{1}{(x-\frac{1}{2})^2 + 1}\,dx$$
$$= \tfrac{1}{4}\arctan(x - \tfrac{1}{2}) + c$$

2. $3 + 4x - 4x^2 = 3 - 4(x^2 - x)$

$$= 3 - 4[(x-\tfrac{1}{2})^2 - \tfrac{1}{4}]$$
$$= 3 - 4(x-\tfrac{1}{2})^2 + 1$$
$$= 4 - 4(x-\tfrac{1}{2})^2$$

$$\int_0^1 \frac{1}{\sqrt{3+4x-4x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{4-4(x-\frac{1}{2})^2}}\,dx$$
$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-(x-\frac{1}{2})^2}}\,dx$$
$$= \tfrac{1}{2}\left[\arcsin(x-\tfrac{1}{2})\right]_0^1$$
$$= \tfrac{1}{2}(\arcsin\tfrac{1}{2} - \arcsin(-\tfrac{1}{2}))$$
$$= \tfrac{1}{2}(\tfrac{\pi}{6} - (-\tfrac{\pi}{6}))$$
$$= \tfrac{\pi}{6}$$

3. (i) $$\int \frac{3x+2}{1+4x^2}\,dx = \int\left(\frac{3x}{1+4x^2} + \frac{2}{4(\frac{1}{4}+x^2)}\right)dx$$
$$= 3\times\tfrac{1}{8}\ln(1+4x^2) + \tfrac{1}{2}\times\tfrac{1}{1/2}\arctan\left(\frac{x}{1/2}\right) + c$$
$$= \tfrac{3}{8}\ln(1+4x^2) + \arctan(2x) + c$$

(ii) $$\int \frac{3x+2}{\sqrt{1-4x^2}}\,dx = \int\left(\frac{3x}{\sqrt{1-4x^2}} + \frac{1}{\sqrt{\frac{1}{4}-x^2}}\right)dx$$
$$= 3\times 2\times -\tfrac{1}{8}\left(1-4x^2\right)^{\frac{1}{2}} + \arcsin\left(\frac{x}{1/2}\right) + c$$
$$= -\tfrac{3}{4}\sqrt{1-4x^2} + \arcsin(2x) + c$$

(iii) $$\frac{3x+2}{(1+4x^2)(1-x)} = \frac{Ax+B}{1+4x^2} + \frac{C}{1-x}$$

$3x + 2 = (1-x)(Ax+B) + C(1+4x^2)$

When $x = 1$, $5 = 5C \quad \Rightarrow C = 1$

When $x = 0$, $2 = B + C \quad \Rightarrow B = 1$

Equating coefficients of x^2: $0 = -A + 4 \quad \Rightarrow A = 4$

$$\int \frac{3x+2}{(1+4x^2)(1-x)}\,dx = \int \left(\frac{4x}{1+4x^2} + \frac{1}{1+4x^2} + \frac{1}{1-x} \right) dx$$
$$= \int \left(\frac{4x}{1+4x^2} + \frac{1}{4}\left(\frac{1}{\frac{1}{4}+x^2} \right) + \frac{1}{1-x} \right) dx$$
$$= \tfrac{1}{2}\ln(1+4x^2) + \tfrac{1}{4} \times \frac{1}{\frac{1}{2}} \arctan\left(\frac{x}{\frac{1}{2}} \right) - \ln|1-x| + c$$
$$= \ln\left(\frac{\sqrt{1+4x^2}}{|1-x|} \right) + \tfrac{1}{2}\arctan(2x) + c$$

4. (i) $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{(4-9x^2)^{\frac{3}{2}}}\,dx$

Let $x = \frac{2}{3}\sin u \quad \Rightarrow \frac{dx}{du} = \frac{2}{3}\cos u \quad \Rightarrow dx = \frac{2}{3}\cos u\, du$

When $x = -\frac{1}{3}$, $\quad -\frac{1}{3} = \frac{2}{3}\sin u \quad \Rightarrow \sin u = -\frac{1}{2} \quad \Rightarrow u = -\frac{\pi}{6}$

When $x = \frac{1}{3}$, $\quad \frac{1}{3} = \frac{2}{3}\sin u \quad \Rightarrow \sin u = \frac{1}{2} \quad \Rightarrow u = \frac{\pi}{6}$

$$\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{(4-9x^2)^{\frac{3}{2}}}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{(4-9\times\frac{4}{9}\sin^2 u)^{\frac{3}{2}}} \times \tfrac{2}{3}\cos u\, du$$
$$= \tfrac{2}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{(4-4\sin^2 u)^{\frac{3}{2}}}\,du$$
$$= \tfrac{2}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{8(1-\sin^2 u)^{\frac{3}{2}}}\,du$$
$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{(1-\sin^2 u)^{\frac{3}{2}}}\,du$$
$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{\cos^3 u}\,du$$
$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sec^2 u\, du$$
$$= \left[\tfrac{1}{12}\tan u\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \tfrac{1}{12}\left(\tfrac{1}{\sqrt{3}} - (-\tfrac{1}{\sqrt{3}}) \right)$$
$$= \frac{1}{6\sqrt{3}}$$

(ii) Let $x = \frac{3}{2}\tan u \quad \Rightarrow \frac{dx}{du} = \frac{3}{2}\sec^2 u \quad \Rightarrow dx = \frac{3}{2}\sec^2 u\, du$

When $x = \frac{3}{2}$, $\tan u = 1 \quad \Rightarrow u = \frac{\pi}{4}$

$$\int_0^{\frac{3}{2}} \frac{1}{(9+4x^2)^{\frac{3}{2}}}\,dx = \int_0^{\pi/4} \frac{1}{(9+9\tan^2 u)^{\frac{3}{2}}} \times \tfrac{3}{2}\sec^2 u\,du$$

$$= \int_0^{\pi/4} \frac{1}{(9\sec^2 u)^{\frac{3}{2}}} \times \tfrac{3}{2}\sec^2 u\,du$$

$$= \int_0^{\pi/4} \frac{1}{27\sec^3 u} \times \tfrac{3}{2}\sec^2 u\,du$$

$$= \int_0^{\pi/4} \frac{1}{18\sec u}\,du$$

$$= \frac{1}{18}\int_0^{\pi/4} \cos u\,du$$

$$= \frac{1}{18}\left[\sin u\right]_0^{\pi/4}$$

$$= \frac{1}{18\sqrt{2}}$$

5. (i) $\dfrac{d}{dx}\left(\ln|\sec x + \tan x|\right) = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$

$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$= \sec x$$

So $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

(ii) (a) $I_1 = \displaystyle\int \frac{1}{(1+a^2x^2)}\,dx = \frac{1}{a^2}\int \frac{1}{(\frac{1}{a^2}+x^2)}\,dx$

$$= \frac{1}{a^2} \times \frac{1}{\frac{1}{a}}\arctan\left(\frac{x}{\frac{1}{a}}\right) + c$$

$$= \frac{1}{a}\arctan(ax) + c$$

(b) $I_{\frac{1}{2}} = \displaystyle\int \frac{1}{(1+a^2x^2)^{\frac{1}{2}}}\,dx$

Let $x = \dfrac{1}{a}\tan u \quad \Rightarrow \dfrac{dx}{du} = \dfrac{1}{a}\sec^2 u$

$$I_{\frac{1}{2}} = \int \frac{1}{(1+\tan^2 u)^{\frac{1}{2}}} \times \frac{1}{a}\sec^2 u \, du$$
$$= \int \frac{1}{\sec u} \times \frac{1}{a}\sec^2 u \, du$$
$$= \frac{1}{a}\int \sec u \, du$$
$$= \frac{1}{a}\ln|\sec u + \tan u| + c$$
$$= \frac{1}{a}\ln\left|\sqrt{1+\tan^2 u} + \tan u\right| + c$$
$$= \frac{1}{a}\ln\left|\sqrt{1+a^2x^2} + ax\right| + c$$

(c) $$I_{\frac{3}{2}} = \int \frac{1}{(1+a^2x^2)^{\frac{3}{2}}} dx$$

Let $x = \frac{1}{a}\tan u \quad \Rightarrow \frac{dx}{du} = \frac{1}{a}\sec^2 u$

$$I_{\frac{3}{2}} = \int \frac{1}{(1+\tan^2 u)^{\frac{3}{2}}} \times \frac{1}{a}\sec^2 u \, du$$
$$= \int \frac{1}{\sec^3 u} \times \frac{1}{a}\sec^2 u \, du$$
$$= \frac{1}{a}\int \frac{1}{\sec u} du$$
$$= \frac{1}{a}\int \cos u \, du$$
$$= \frac{1}{a}\sin u + c$$
$$= \frac{1}{a} \times \frac{ax}{\sqrt{1+a^2x^2}} + c$$
$$= \frac{x}{\sqrt{1+a^2x^2}} + c$$