Section 3: Further integration

Solutions to Exercise level 1

1. (i) $2x - x^2 = -(x^2 - 2x)$

$$= -[(x-1)^2 - 1]$$

$$= 1 - (x-1)^2$$

$$\int \frac{1}{\sqrt{2x - x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$

$$= \arcsin(x-1) + c$$

(ii) $x^2 + 4x + 5 = (x+2)^2 - 4 + 5$

$$= 1 + (x+2)^2$$

$$\int \frac{1}{x^2 + 4x + 5}\,dx = \int \frac{1}{1+(x+2)^2}\,dx$$

$$= \arctan(x+2) + c$$

2. (i) $\dfrac{8}{(x+2)(x^2+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+4}$

$8 = A(x^2+4) + (Bx+C)(x+2)$

Putting $x = -2 \Rightarrow 8 = 8A \Rightarrow A = 1$

Putting $x = 0 \Rightarrow 8 = 4A + 2C \Rightarrow 8 = 4 + 2C \Rightarrow C = 2$

Equating coefficients of $x^2 \Rightarrow 0 = A + B \Rightarrow B = -A = -1$

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{2-x}{x^2+4}$$

(ii) $\displaystyle\int_0^2 \frac{8}{(x+2)(x^2+4)}\,dx = \int_0^2 \left(\frac{1}{x+2} - \frac{x}{x^2+4} + \frac{2}{x^2+4}\right)dx$

$$= \left[\ln|x+2| - \tfrac{1}{2}\ln(x^2+4) + 2\times\tfrac{1}{2}\arctan\tfrac{x}{2}\right]_0^2$$

$$= \ln 4 - \tfrac{1}{2}\ln 8 + \arctan 1 - (\ln 2 - \tfrac{1}{2}\ln 4 + \arctan 0)$$

$$= 2\ln 2 - \tfrac{3}{2}\ln 2 + \tfrac{1}{4}\pi - \ln 2 + \tfrac{1}{2}\times 2\ln 2$$

$$= \tfrac{1}{2}\ln 2 + \tfrac{1}{4}\pi$$

3. (i) $\displaystyle\int \sqrt{1 - 4x^2}\,dx$

Let $x = \tfrac{1}{2}\sin u \Rightarrow \dfrac{dx}{du} = \tfrac{1}{2}\cos u \Rightarrow dx = \tfrac{1}{2}\cos u\,du$

$$\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-4\times\tfrac{1}{4}\sin^2 u}\times\tfrac{1}{2}\cos u\,du$$
$$= \int \tfrac{1}{2}\sqrt{1-\sin^2 u}\times\cos u\,du$$
$$= \int \tfrac{1}{2}\cos^2 u\,du$$
$$= \int \tfrac{1}{4}(\cos 2u + 1)\,du$$
$$= \tfrac{1}{4}\times\tfrac{1}{2}\sin 2u + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\sin u\cos u + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\sin u\sqrt{1-\sin^2 u} + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\times 2x\sqrt{1-(2x)^2} + \tfrac{1}{4}\arcsin 2x + c$$
$$= \tfrac{1}{2}x\sqrt{1-4x^2} + \tfrac{1}{4}\arcsin 2x + c$$

(ii) $\displaystyle\int_0^2 \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$

Let $x = 2\tan u \quad \Rightarrow \dfrac{dx}{du} = 2\sec^2 u \quad \Rightarrow dx = 2\sec^2 u\,du$

When $x = 0$, $0 = 2\tan u \quad \Rightarrow u = 0$

When $x = 2$, $2 = 2\tan u \quad \Rightarrow \tan u = 1 \quad \Rightarrow u = \frac{\pi}{4}$

$$\int_0^2 \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{(4+4\tan^2 u)^{\frac{3}{2}}}\times 2\sec^2 u\,du$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{8(1+\tan^2 u)^{\frac{3}{2}}}\times 2\sec^2 u\,du$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2 u)^{\frac{3}{2}}}\times\sec^2 u\,du$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^3 u}\times\sec^2 u\,du$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\sec u}\,du$$
$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{4}\cos u\,du$$
$$= \left[\tfrac{1}{4}\sin u\right]_0^{\frac{\pi}{4}}$$
$$= \tfrac{1}{4}(\sin\tfrac{\pi}{4} - \sin 0)$$
$$= \frac{1}{4\sqrt{2}}$$