Section 2: Mean values and general integration

Exercise level 2

1. Mean value of $y = x^2$ is $\frac{1}{a}\int_0^a x^2 dx = \frac{1}{a}\left[\frac{1}{3}x^3\right]_0^a$

Mean value of $y = x^3$ is $\frac{1}{a}\int_0^a x^3 dx = \frac{1}{a}\left[\frac{1}{4}x^4\right]_0^a$

$$\frac{1}{a}\left[\tfrac{1}{3}x^3\right]_0^a = \frac{1}{a}\left[\tfrac{1}{4}x^4\right]_0^a$$

$$\frac{1}{a}\times\frac{a^3}{3} = \frac{1}{a}\times\frac{a^4}{4}$$

$$4a^2 = 3a^3$$

$$a^2(4-3a) = 0$$

$$a = \tfrac{4}{3}$$

2. (i)

$$\int_1^{24}\frac{4}{\sqrt{9+16x^2}}dx = \int_1^{24}\frac{4}{4\sqrt{\frac{9}{16}+x^2}}dx$$

$$= \int_1^{24}\frac{1}{\sqrt{\frac{9}{16}+x^2}}dx$$

$$= \left[\operatorname{arsinh}\frac{x}{3/4}\right]_1^{24} = \left[\operatorname{arsinh}\frac{4x}{3}\right]_1^{24}$$

$$= \operatorname{arsinh}32 - \operatorname{arsinh}\tfrac{4}{3} = 3.061$$

(ii)

$$\int_1^2\frac{9}{\sqrt{16-4x^2}}dx = \int_1^2\frac{9}{2\sqrt{4-x^2}}dx$$

$$= \left[\frac{9}{2}\arcsin\frac{x}{2}\right]_1^2$$

$$= \tfrac{9}{2}(\arcsin 1 - \arcsin\tfrac{1}{2})$$

$$= \tfrac{9}{2}(\tfrac{1}{2}\pi - \tfrac{1}{6}\pi)$$

$$= \tfrac{3}{2}\pi = 4.712$$

(iii)

$$\int_1^2\frac{16}{\sqrt{9x^2-4}}dx = \int_1^2\frac{16}{3\sqrt{x^2-\frac{4}{9}}}dx$$

$$= \left[\frac{16}{3}\operatorname{arcosh}\left(\frac{x}{2/3}\right)\right]_1^2 = \left[\frac{16}{3}\operatorname{arcosh}\left(\frac{3x}{2}\right)\right]_1^2$$

$$= \tfrac{16}{3}(\operatorname{arcosh}3 - \operatorname{arcosh}\tfrac{3}{2}) = 4.268$$

so (ii) is the biggest.

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3. (i) $\int_0^2 \frac{1}{3x^2+4}\,dx = \frac{1}{3}\int_0^2 \frac{1}{x^2+\frac{4}{3}}\,dx$

$$= \left[\tfrac{1}{3}\times\frac{1}{\frac{2}{\sqrt{3}}}\arctan\left(\frac{x}{\frac{2}{\sqrt{3}}}\right)\right]_0^2$$

$$= \left[\frac{\sqrt{3}}{6}\arctan\left(\frac{x\sqrt{3}}{2}\right)\right]_0^2$$

$$= \frac{\sqrt{3}}{6}\left(\arctan\sqrt{3} - \arctan 0\right)$$

$$= \frac{\sqrt{3}}{6}\times\frac{\pi}{3}$$

$$= \frac{\pi\sqrt{3}}{18}$$

(ii) $x\sqrt{3} = 2\tan\theta$

$$\frac{dx}{d\theta} = \frac{2}{\sqrt{3}}\sec^2\theta$$

When $x = 2$, $\tan\theta = \sqrt{3} \quad \Rightarrow \theta = \frac{\pi}{3}$

When $x = 0$, $\tan\theta = 0 \quad \Rightarrow \theta = 0$

$$\int_0^2 \frac{1}{(3x^2+4)^{\frac{3}{2}}}\,dx = \int_0^{\pi/3}\frac{1}{(4\tan^2\theta+4)^{\frac{3}{2}}}\times\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3}\frac{1}{4^{\frac{3}{2}}(\sec^2\theta)^{\frac{3}{2}}}\times\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3}\frac{1}{8\sec^3\theta}\times\frac{2}{\sqrt{3}}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/3}\frac{1}{4\sqrt{3}}\cos\theta\,d\theta$$

$$= \left[\frac{1}{4\sqrt{3}}\sin\theta\right]_0^{\pi/3}$$

$$= \frac{1}{4\sqrt{3}}(\sin\tfrac{\pi}{3} - \sin 0)$$

$$= \frac{1}{4\sqrt{3}}\times\frac{\sqrt{3}}{2}$$

$$= \frac{1}{8}$$

4. (i) $\int\sqrt{1-4x^2}\,dx$

Let $x = \frac{1}{2}\sin u \quad \Rightarrow \frac{dx}{du} = \frac{1}{2}\cos u \quad \Rightarrow dx = \frac{1}{2}\cos u\,du$

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$$\int \sqrt{1-4x^2}\,dx = \int \sqrt{1-4\times\tfrac{1}{4}\sin^2 u}\times\tfrac{1}{2}\cos u\,du$$
$$= \int \tfrac{1}{2}\sqrt{1-\sin^2 u}\times\cos u\,du$$
$$= \int \tfrac{1}{2}\cos^2 u\,du$$
$$= \int \tfrac{1}{4}(\cos 2u + 1)\,du$$
$$= \tfrac{1}{4}\times\tfrac{1}{2}\sin 2u + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\sin u\cos u + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\sin u\sqrt{1-\sin^2 u} + \tfrac{1}{4}u + c$$
$$= \tfrac{1}{4}\times 2x\sqrt{1-(2x)^2} + \tfrac{1}{4}\arcsin 2x + c$$
$$= \tfrac{1}{2}x\sqrt{1-4x^2} + \tfrac{1}{4}\arcsin 2x + c$$

(ii) $\int \sqrt{x^2-9}\,dx$

Let $x = 3\cosh u \quad \Rightarrow \dfrac{dx}{du} = 3\sinh u \quad \Rightarrow dx = 3\sinh u\,du$

$$\int \sqrt{x^2-9}\,dx = \int \sqrt{9\cosh^2 u - 9}\times 3\sinh u\,du$$
$$= \int 3\sinh u \times 3\sinh u\,du$$
$$= \int 9\sinh^2 u\,du$$
$$= \int \tfrac{9}{2}(\cosh 2u - 1)\,du$$
$$= \tfrac{9}{2}(\tfrac{1}{2}\sinh 2u - u) + c$$
$$= \tfrac{9}{2}\sinh u\cosh u - \tfrac{9}{2}u + c$$
$$= \tfrac{9}{2}\times\sqrt{\tfrac{x^2}{9}-1}\times\tfrac{x}{3} - \tfrac{9}{2}\cosh^{-1}\tfrac{x}{3} + c$$
$$= \tfrac{1}{2}x\sqrt{x^2-9} - \tfrac{9}{2}\cosh^{-1}\tfrac{x}{3} + c$$

(iii) $\displaystyle\int_0^2 \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx$

Let $x = 2\tan u \quad \Rightarrow \dfrac{dx}{du} = 2\sec^2 u \quad \Rightarrow dx = 2\sec^2 u\,du$

When $x = 0$, $0 = 2\tan u \quad \Rightarrow u = 0$

When $x = 2$, $2 = 2\tan u \quad \Rightarrow \tan u = 1 \quad \Rightarrow u = \frac{\pi}{4}$

$$\int_0^2 \frac{1}{(4+x^2)^{\frac{3}{2}}}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{(4+4\tan^2 u)^{\frac{3}{2}}} \times 2\sec^2 u\,du$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{8(1+\tan^2 u)^{\frac{3}{2}}} \times 2\sec^2 u\,du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2 u)^{\frac{3}{2}}} \times \sec^2 u\,du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\sec^3 u} \times \sec^2 u\,du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{1}{\sec u}\,du$$

$$= \int_0^{\frac{\pi}{4}} \tfrac{1}{4}\cos u\,du$$

$$= \left[\tfrac{1}{4}\sin u\right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{4}(\sin\tfrac{\pi}{4} - \sin 0)$$

$$= \frac{1}{4\sqrt{2}}$$

(iv) $\displaystyle\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{(4-9x^2)^{\frac{3}{2}}}\,dx$

Let $x = \tfrac{2}{3}\sin u \quad \Rightarrow \dfrac{dx}{du} = \tfrac{2}{3}\cos u \quad \Rightarrow dx = \tfrac{2}{3}\cos u\,du$

When $x = -\tfrac{1}{3}, \quad -\tfrac{1}{3} = \tfrac{2}{3}\sin u \quad \Rightarrow \sin u = -\tfrac{1}{2} \quad \Rightarrow u = -\tfrac{\pi}{6}$

When $x = \tfrac{1}{3}, \quad \tfrac{1}{3} = \tfrac{2}{3}\sin u \quad \Rightarrow \sin u = \tfrac{1}{2} \quad \Rightarrow u = \tfrac{\pi}{6}$

$$\int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{(4-9x^2)^{\frac{3}{2}}}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{(4-9\times\frac{4}{9}\sin^2 u)^{\frac{3}{2}}} \times \tfrac{2}{3}\cos u\,du$$

$$= \tfrac{2}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{(4-4\sin^2 u)^{\frac{3}{2}}}\,du$$

$$= \tfrac{2}{3}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{8(1-\sin^2 u)^{\frac{3}{2}}}\,du$$

$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{(1-\sin^2 u)^{\frac{3}{2}}}\,du$$

$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{\cos u}{\cos^3 u}\,du$$

$$= \tfrac{1}{12}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sec^2 u\,du$$

$$= \left[\tfrac{1}{12}\tan u\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \tfrac{1}{12}\left(\tfrac{1}{\sqrt{3}} - (-\tfrac{1}{\sqrt{3}})\right)$$

$$= \frac{1}{6\sqrt{3}}$$

(v) $\int_{0.5}^{1} \frac{x^3}{\sqrt{4x^2-1}}\,dx$

Let $x = \frac{1}{2}\cosh u \quad \Rightarrow \frac{dx}{du} = \frac{1}{2}\sinh u \quad \Rightarrow dx = \frac{1}{2}\sinh u\,du$

$$
\begin{aligned}
\int_{0.5}^{1} \frac{x^3}{\sqrt{4x^2-1}}\,dx &= \int_{x=0.5}^{x=1} \frac{\frac{1}{8}\cosh^3 u}{\sqrt{\cosh^2 u - 1}} \times \frac{1}{2}\sinh u\,du \\
&= \frac{1}{16}\int_{x=0.5}^{x=1} \cosh^3 u\,du \\
&= \frac{1}{16}\int_{x=0.5}^{x=1} \cosh u(1+\sinh^2 u)\,du \\
&= \frac{1}{16}\int_{x=0.5}^{x=1} (\cosh u + \cosh u \sinh^2 u)\,du \\
&= \frac{1}{16}\left[\sinh u + \frac{1}{3}\sinh^3 u\right]_{x=0.5}^{x=1} \\
&= \frac{1}{16}\left[\sqrt{\cosh^2 u - 1} + \frac{1}{3}(\cosh^2 u - 1)^{\frac{3}{2}}\right]_{x=0.5}^{x=1} \\
&= \frac{1}{16}\left[\sqrt{4x^2-1} + \frac{1}{3}(4x^2-1)^{\frac{3}{2}}\right]_{0.5}^{1} \\
&= \frac{1}{16}\left(\sqrt{3} + \frac{1}{3}\times 3\sqrt{3} - 0\right) \\
&= \frac{1}{8}\sqrt{3}
\end{aligned}
$$