Section 2: Inverse trigonometric functions

Exercise level 3 solutions

1. (i) $\frac{d}{dx}(\arcsin x \times \arccos x) = \frac{1}{\sqrt{1-x^2}}\arccos x - \frac{1}{\sqrt{1-x^2}}\arcsin x$

At stationary point $\frac{1}{\sqrt{1-x^2}}\arccos x = \frac{1}{\sqrt{1-x^2}}\arcsin x$

$\arccos x = \arcsin x$

$x = \frac{1}{\sqrt{2}}$ (when $\arccos x = \arcsin x = \frac{\pi}{4}$)

so maximum value of $(\arcsin x)(\arccos x) = \frac{\pi}{4} \times \frac{\pi}{4} = \frac{\pi^2}{16}$

(ii) $\arcsin x + \arccos x = \frac{\pi}{2}$ = constant

The product of two numbers whose sum is constant is a maximum when the numbers are equal, so $\arcsin x = \arccos x = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$

so maximum value of $(\arcsin x)(\arccos x) = \frac{\pi}{4} \times \frac{\pi}{4} = \frac{\pi^2}{16}$

2. (i) $\int_0^1 \left(\frac{9}{1+16x^2} + \frac{16}{9+x^2} + \frac{1}{16+9x^2} \right) dx$

$= \int_0^1 \left(\frac{\frac{9}{16}}{\frac{1}{16}+x^2} + \frac{16}{9+x^2} + \frac{\frac{1}{9}}{\frac{16}{9}+x^2} \right) dx$

$= \left[\frac{9}{16} \times 4\arctan(4x) + \frac{16}{3}\arctan\left(\frac{x}{3}\right) + \frac{1}{9} \times \frac{3}{4}\arctan\left(\frac{3x}{4}\right) \right]_0^1$

$= \frac{9}{4}\arctan(4) + \frac{16}{3}\arctan\left(\frac{1}{3}\right) + \frac{1}{12}\arctan\left(\frac{3}{4}\right)$

(ii) $\int_0^{\frac{1}{4}} \left(\frac{9}{\sqrt{1-16x^2}} + \frac{16}{\sqrt{9-x^2}} + \frac{1}{\sqrt{16-9x^2}} \right) dx$

$= \int_0^{\frac{1}{4}} \left(\frac{9}{4\sqrt{\frac{1}{16}-x^2}} + \frac{16}{\sqrt{9-x^2}} + \frac{1}{3\sqrt{\frac{16}{9}-x^2}} \right) dx$

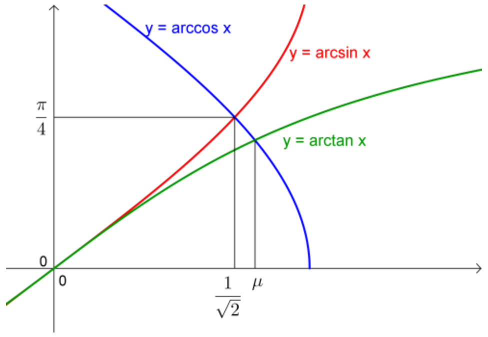
$= \lim_{a \to \frac{1}{4}} \int_0^a \frac{9}{4\sqrt{\frac{1}{16}-x^2}} dx + \int_0^{\frac{1}{4}} \left(\frac{16}{\sqrt{9-x^2}} + \frac{1}{3\sqrt{\frac{16}{9}-x^2}} \right) dx$

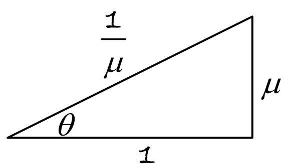
$$= \lim_{a \to \frac{1}{4}} \left[\frac{9}{4} \arcsin(4x) \right]_0^a + \left[16 \arcsin\left(\frac{x}{3}\right) + \frac{1}{3} \arcsin\left(\frac{3x}{4}\right) \right]_0^{\frac{1}{4}}$$

$$= \frac{9}{4} \arcsin 1 + 16 \arcsin\left(\frac{1}{12}\right) + \frac{1}{3} \arcsin\left(\frac{3}{16}\right)$$

$$= \frac{9\pi}{8} + 16 \arcsin\left(\frac{1}{12}\right) + \frac{1}{3} \arcsin\left(\frac{3}{16}\right)$$

3. $\arcsin x = \arccos x \Rightarrow x = \frac{1}{\sqrt{2}}$





$\arccos \mu = \arctan \mu = \theta$

$\Rightarrow \cos\theta = \mu, \quad \tan\theta = \mu$

$$\Rightarrow 1 + \mu^2 = \frac{1}{\mu^2}$$

$$\Rightarrow \mu^4 + \mu^2 - 1 = 0$$

$$\Rightarrow \mu^2 = \frac{-1 + \sqrt{5}}{2} \Rightarrow \mu = 0.78615...$$

$$\text{Area} = \int_0^{1/\sqrt{2}} \arcsin x \, dx + \int_{1/\sqrt{2}}^{\mu} \arccos x \, dx - \int_0^{\mu} \arctan x \, dx$$

$$\int_0^{1/\sqrt{2}} \arcsin x \, dx = [x \arcsin x]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1 - x^2}} dx$$

$$= \left[x \arcsin x + \sqrt{1 - x^2} \right]_0^{1/\sqrt{2}}$$

$$= \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$\int_{1/\sqrt{2}}^{\mu} \arccos x \, dx = [x \arccos x]_{1/\sqrt{2}}^{\mu} + \int_{1/\sqrt{2}}^{\mu} \frac{x}{\sqrt{1 - x^2}} dx$$

$$= \left[x \arccos x - \sqrt{1 - x^2} \right]_{1/\sqrt{2}}^{\mu}$$

$$= \mu \arccos \mu - \sqrt{1 - \mu^2} - \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$$

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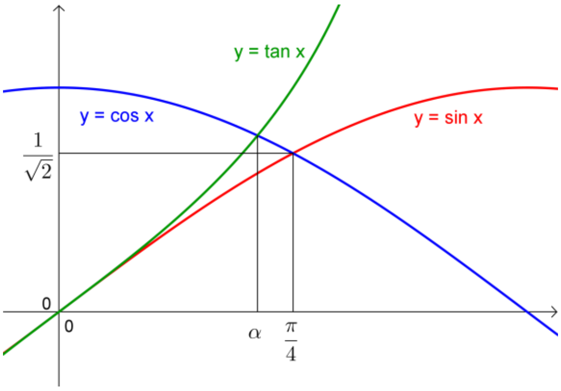
$$\int_0^{\mu} \arctan x \, dx = [x \arctan x]_0^{\mu} - \int_0^{\mu} \frac{x}{1+x^2} dx$$
$$= \left[x \arctan x - \tfrac{1}{2}\ln(1+x^2)\right]_0^{\mu}$$
$$= \mu \arctan \mu - \tfrac{1}{2}\ln(1+\mu^2)$$

$$\text{Area } = \frac{2}{\sqrt{2}} - 1 + \mu \arccos \mu - \sqrt{1-\mu^2} - \mu \arctan \mu + \tfrac{1}{2}\ln(1+\mu^2)$$
$$= 0.0368 \text{ (3 s.f.)}$$

<u>Alternative method:</u>

The area enclosed by $\arcsin x$, $\arccos x$ and $\arctan x$ is the same as that enclosed by $\sin x$, $\cos x$ and $\tan x$ (the area is reflected in the line $y = x$).



$$\tan \alpha = \cos \alpha \Rightarrow \sin \alpha = \cos^2 \alpha$$
$$\Rightarrow \sin \alpha = 1 - \sin^2 \alpha$$
$$\Rightarrow \sin^2 \alpha + \sin \alpha - 1 = 0$$
$$\Rightarrow \sin \alpha = \frac{-1+\sqrt{5}}{2}$$

$$\text{Area } = \int_0^{\alpha} \tan x \, dx + \int_{\alpha}^{\pi/4} \cos x \, dx - \int_0^{\pi/4} \sin x \, dx$$
$$= [-\ln|\cos x|]_0^{\alpha} + [\sin x]_{\alpha}^{\pi/4} + [\cos x]_0^{\pi/4}$$
$$= -\ln(\cos \alpha) + \frac{1}{\sqrt{2}} - \sin \alpha + \frac{1}{\sqrt{2}} - 1$$
$$= 0.0368 \text{ (3 s.f.)}$$