Section 2: Inverse trigonometric functions

Exercise level 2 solutions

1. (i) $\frac{d}{dx}(\arctan x^2) = 2x \times \frac{1}{1+(x^2)^2}$

$= \frac{2x}{1+x^4}$

(ii) $\frac{d}{dx}\left(\arcsin \frac{1}{x}\right) = -\frac{1}{x^2} \times \frac{1}{\sqrt{1-(\frac{1}{x})^2)}}$

$= -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$= -\frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$

$= -\frac{1}{x\sqrt{x^2-1}}$

(iii) $\frac{d}{dx}(\arccos(1-\sqrt{x})) = -\frac{1}{2}x^{-\frac{1}{2}} \times -\frac{1}{\sqrt{1-(1-\sqrt{x})^2)}}$

$= \frac{1}{2\sqrt{x}\sqrt{1-(1-2\sqrt{x}+x)}}$

$= \frac{1}{2\sqrt{x}\sqrt{2\sqrt{x}-x}}$

(iv) $\frac{d}{dx}(\arcsin(\arctan x)) = \frac{1}{\sqrt{1-(\arctan x)^2}} \times \frac{d}{dx}(\arctan x)$

$= \frac{1}{\sqrt{1-(\arctan x)^2}} \times \frac{1}{1+x^2}$

$= \frac{1}{(1+x^2)\sqrt{1-(\arctan x)^2}}$

2. (a) $\int_c^d \frac{9}{1+4x^2}dx = \frac{1}{4}\int_c^d \frac{9}{\frac{1}{4}+x^2}dx = \frac{9}{4}\left[\frac{1}{\frac{1}{2}}\arctan\left(\frac{x}{\frac{1}{2}}\right)\right]_c^d = \frac{9}{2}[\arctan(2x)]_c^d$

(i) $\int_0^{1/2} \frac{9}{1+4x^2}dx = \frac{9}{2}[\arctan(2x)]_0^{1/2} = \frac{9}{2}\arctan 1 - 0$

$= \frac{9\pi}{8}$

(ii) $\int_{\sqrt{3}/2}^{\infty} \frac{9}{1+4x^2}\,dx = \frac{9}{2}\left[\arctan(2x)\right]_{\sqrt{3}/2}^{\infty}$

$$= \frac{9}{2}\left(\arctan\infty - \arctan\sqrt{3}\right) = \frac{9}{2}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)$$

$$= \frac{3\pi}{4}$$

(b) $\int_c^d \frac{9}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int_c^d \frac{9}{\sqrt{\frac{1}{4}-x^2}}\,dx = \left[\frac{9}{2}\arcsin\left(\frac{x}{\frac{1}{2}}\right)\right]_c^d$

$$= \left[\frac{9}{2}\arcsin(2x)\right]_c^d$$

(i) $\int_0^{1/2} \frac{9}{\sqrt{1-4x^2}}\,dx = \left[\frac{9}{2}\arcsin(2x)\right]_0^{1/2} = \frac{9}{2}\arcsin 1 - 0$

$$= \frac{9\pi}{4}$$

(ii) $\int_{1/2\sqrt{2}}^{\sqrt{3}/4} \frac{9}{\sqrt{1-4x^2}}\,dx = \left[\frac{9}{2}\arcsin(2x)\right]_{1/2\sqrt{2}}^{\sqrt{3}/4}$

$$= \frac{9}{2}\left(\arcsin\frac{\sqrt{3}}{2} - \arcsin\frac{1}{\sqrt{2}}\right) = \frac{9}{2}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$= \frac{3\pi}{8}$$

3. (i) $\int \arcsin 2x\,dx = \int 1\times\arcsin 2x\,dx$

Let $u = \arcsin 2x \quad \Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$

$\frac{dv}{dx} = 1 \quad \Rightarrow v = x$

Using integration by parts, $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int 1\times\arcsin 2x\,dx = x\arcsin 2x - \int x\times\frac{2}{\sqrt{1-4x^2}}\,dx$$

$$= x\arcsin 2x - \int \frac{2x}{\sqrt{1-4x^2}}\,dx$$

$$= x\arcsin 2x + \tfrac{1}{4}\int -8x(1-4x^2)^{-\frac{1}{2}}\,dx$$

$$= x\arcsin 2x + \tfrac{1}{4}\times 2(1-4x^2)^{\frac{1}{2}} + c$$

$$= x\arcsin 2x + \tfrac{1}{2}\sqrt{1-4x^2} + c$$

(ii) $\int \arctan 3x\,dx = \int 1\times\arctan 3x\,dx$

Let $u = \arctan 3x \quad \Rightarrow \frac{du}{dx} = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$

$\frac{dv}{dx} = 1 \quad \Rightarrow v = x$

Using integration by parts, $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$

$$\int \arctan 3x\,dx = x\arctan 3x - \int \frac{3x}{1+9x^2}\,dx$$

$$= x\arctan 3x - \tfrac{1}{6}\ln(1+9x^2) + c$$

(iii) $\int 1\times(\arcsin x + \arctan x)\,dx$

$$= x(\arcsin x + \arctan x) - \int\left(\frac{x}{\sqrt{1-x^2}} + \frac{x}{1+x^2}\right)dx$$

$$= x(\arcsin x + \arctan x) + (1-x^2)^{\frac{1}{2}} - \tfrac{1}{2}\ln(1+x^2) + c$$