Section 2: The inverse trigonometric functions

Solutions to Exercise level 1

1. $y = \arccos x$

$\cos y = x$

Differentiating implicitly: $-\sin y \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}$$

2. (i) $$\frac{d}{dx}(\arcsin 3x) = 3 \times \frac{1}{\sqrt{1-(3x)^2)}}$$
$$= \frac{3}{\sqrt{1-9x^2}}$$

(ii) $$\frac{d}{dx}(\arctan 2x) = 2 \times \frac{1}{1+(2x)^2}$$
$$= \frac{2}{1+4x^2}$$

(iii) $$\frac{d}{dx}(\arccos 4x) = 4 \times -\frac{1}{\sqrt{1-(4x)^2)}}$$
$$= -\frac{4}{\sqrt{1-16x^2}}$$

(iv) $$\frac{d}{dx}(\arcsin(4x+1)) = 4 \times \frac{1}{\sqrt{1-(4x+1)^2)}}$$
$$= \frac{4}{\sqrt{1-(4x+1)^2}}$$

(v) $$\frac{d}{dx}(\arctan(10-3x)) = -3 \times \frac{1}{1+(10-3x)^2}$$
$$= \frac{-3}{1+(10-3x)^2}$$

(vi) $$\frac{d}{dx}(\arccos(5x+2)) = 5 \times -\frac{1}{\sqrt{1-(5x+2)^2}}$$
$$= -\frac{5}{\sqrt{1-(5x+2)^2}}$$

3. (i) Using the standard result $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+c$ with $a=2$:

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \arcsin\frac{x}{2}+c$$

(ii) Using the standard result $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+c$ with $a=\sqrt{3}$:

$$\int \frac{1}{3+x^2}\,dx = \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}+c$$

(iii) $\int \frac{2}{1+25x^2}\,dx = \frac{2}{25}\int \frac{1}{\frac{1}{25}+x^2}\,dx$

Using the standard result $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+c$ with $a=\frac{1}{5}$:

$$\begin{aligned}\int \frac{2}{1+25x^2}\,dx &= \frac{2}{25}\int \frac{1}{\frac{1}{25}+x^2}\,dx \\ &= \frac{2}{25}\times 5\arctan 5x+c \\ &= \tfrac{2}{5}\arctan 5x+c\end{aligned}$$

(iv) $\int \frac{4}{\sqrt{16-9x^2}}\,dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{16}{9}-x^2}}\,dx$

Using the standard result $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+c$ with $a=\frac{4}{3}$:

$$\int \frac{4}{\sqrt{16-9x^2}}\,dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{16}{9}-x^2}}\,dx = \tfrac{4}{3}\arcsin\tfrac{3x}{4}+c$$

4. (i) Using the standard result $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a}+c$ with $a=3$:

$$\begin{aligned}\int_0^3 \frac{1}{9+x^2}\,dx &= \left[\frac{1}{3}\arctan\frac{x}{3}\right]_0^3 \\ &= \tfrac{1}{3}(\arctan 1-\arctan 0) \\ &= \tfrac{1}{3}(\tfrac{\pi}{4}-0) \\ &= \tfrac{1}{12}\pi\end{aligned}$$

(ii) $\int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{1}{\sqrt{1-16x^2}}\,dx = \frac{1}{4}\int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{1}{\sqrt{\frac{1}{16}-x^2}}\,dx$

Using the standard result $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a}+c$ with $a=\frac{1}{4}$:

$$\int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{1}{\sqrt{1-16x^2}}\,dx = \frac{1}{4}\int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{1}{\sqrt{\frac{1}{16}-x^2}}\,dx$$
$$= \tfrac{1}{4}\left[\arcsin 4x\right]_{-\frac{1}{8}}^{\frac{1}{8}}$$
$$= \tfrac{1}{4}\left(\arcsin\tfrac{1}{2} - \arcsin\left(-\tfrac{1}{2}\right)\right)$$
$$= \tfrac{1}{4}\left(\tfrac{\pi}{6} - \left(-\tfrac{\pi}{6}\right)\right)$$
$$= \tfrac{1}{12}\pi$$

(iii) $\displaystyle\int_0^{\frac{2}{\sqrt{3}}} \frac{1}{4+9x^2}\,dx = \frac{1}{9}\int_0^{\frac{2}{\sqrt{3}}} \frac{1}{\frac{4}{9}+x^2}\,dx$

Using the standard result $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + c$ with $a = \frac{2}{3}$:

$$\int_0^{\frac{2}{\sqrt{3}}} \frac{1}{4+9x^2}\,dx = \frac{1}{9}\int_0^{\frac{2}{\sqrt{3}}} \frac{1}{\frac{4}{9}+x^2}\,dx$$
$$= \tfrac{1}{9}\left[\tfrac{3}{2}\arctan\tfrac{3}{2}x\right]_0^{\frac{2}{\sqrt{3}}}$$
$$= \tfrac{1}{6}\left(\arctan\sqrt{3} - \arctan 0\right)$$
$$= \tfrac{1}{6}\times\tfrac{\pi}{3}$$
$$= \tfrac{1}{18}\pi$$

(iv) $\displaystyle\int_0^{1.25} \frac{1}{\sqrt{25-4x^2}}\,dx = \frac{1}{2}\int_0^{1.25} \frac{1}{\sqrt{\frac{25}{4}-x^2}}\,dx$

Using the standard result $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + c$ with $a = \frac{5}{2}$:

$$\int_0^{1.25} \frac{1}{\sqrt{25-4x^2}}\,dx = \frac{1}{2}\int_0^{1.25} \frac{1}{\sqrt{\frac{25}{4}-x^2}}\,dx$$
$$= \tfrac{1}{2}\left[\arcsin\tfrac{2}{5}x\right]_0^{1.25}$$
$$= \tfrac{1}{2}\left(\arcsin\tfrac{1}{2} - \arcsin 0\right)$$
$$= \tfrac{1}{2}\times\tfrac{\pi}{6}$$
$$= \tfrac{1}{12}\pi$$