Section 1: Finding and using Maclaurin series

Solutions to Exercise level 2

1. $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$

$$e^{1/3} = 1 + \frac{1}{3} + \frac{(1/3)^2}{2} + \frac{(1/3)^3}{6} + \frac{(1/3)^4}{24} + \dots$$

$$= 1 + \frac{1}{3} + \frac{1}{18} + \frac{1}{162} + \frac{1}{1944} + \dots$$

$$= 1.395576132\dots$$

Next term is $\frac{(1/3)^5}{120} = 0.000034$

so 5.s.f. is sensible, giving 1.3956

2. $y = \frac{2e^x + x}{e^x} = 2 + xe^{-x}$

$$= 2 + x\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots\right)$$

$$= 2 + x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + ..$$

3. $y = e^x \cos x$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\dots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\dots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots - \frac{x^2}{2} - \frac{x^3}{2} - \frac{x^4}{4} + \dots + \frac{x^4}{24}\dots$$

$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \dots$$

4. (i) $\cos\theta = 1 - \frac{\theta^2}{2} + \dots$

$$\sin\theta = \theta - \frac{\theta^3}{6} + \dots$$

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(ii) (a) $\sin\theta = 2\theta - \dfrac{(2\theta)^3}{6} + \ldots$

$$= 2\theta - \frac{8\theta^3}{6} + \ldots$$

$$= 2\theta - \frac{4\theta^3}{3} + \ldots$$

(b) $\sin 2\theta = 2\sin\theta\cos\theta$

$$= 2\left(\theta - \frac{\theta^3}{6} + \ldots\right)\left(1 - \frac{\theta^2}{2} + \ldots\right)$$

$$= 2\left(\theta - \frac{\theta^3}{6} - \frac{\theta^3}{2} + \ldots\right)$$

$$= 2\theta - \frac{4\theta^3}{3}$$

5. (i) $y = (1-3x)^{-3}$

$y' = -3(1-3x)^{-4} \times -3 = 9(1-3x)^{-4}$

$y'' = -4 \times 9(1-3x)^{-5} \times -3 = 108(1-3x)^{-5}$

$y'' = -5 \times 108(1-3x)^{-6} \times -3 = 1620(1-3x)^{-6}$

(ii) $y(0) = 1$

$y'(0) = 9$

$y''(0) = 108$

$y''(0) = 1620$

$$y = 1 + 9x + \frac{108x^2}{2} + \frac{1620x^3}{6} + \ldots$$

$$= 1 + 9x + 54x^2 + 270x^3 + \ldots$$

(iii) Valid for $-\frac{1}{3} < x < \frac{1}{3}$.

6. (i) $f'(x) = \dfrac{1}{1+(x^3)^2} \times 3x^2 = \dfrac{3x^2}{1+x^6}$

$f'(x) = 3x^2\left(1+x^6\right)^{-1}$

$$= 3x^2(1 - x^6 + x^{12} - x^{18} + \ldots)$$

$$= 3x^2 - 3x^8 + 3x^{14} - 3x^{20} + \ldots$$

$f(x) = \int (3x^2 - 3x^8 + 3x^{14} - 3x^{20} + \ldots)dx$

$$= c + x^3 - \tfrac{1}{3}x^9 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^{21} + \ldots$$

When $x = 0$, $\arctan(x^3) = 0$, so $c = 0$

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$$f(x) = x^3 - \tfrac{1}{3}x^9 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^{21} + \ldots$$

(ii) $$\arctan(x^3) = x^3 - \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} - \frac{(x^3)^7}{7} + \ldots.$$

$$= x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \ldots.$$

Valid for $|x| \le 1$

7. (i) $$f(x) = \arcsin\left(\tfrac{3}{5} + x\right)$$

$$f'(x) = \frac{1}{\sqrt{1 - \left(\tfrac{3}{5} + x\right)^2}}$$

$$f''(x) = -\tfrac{1}{2}\left(1 - \left(\tfrac{3}{5} + x\right)^2\right)^{-\frac{3}{2}} \times -2(\tfrac{3}{5} + x)$$

$$= \left(\tfrac{3}{5} + x\right)\left(1 - (\tfrac{3}{5} + x)^2\right)^{-\frac{3}{2}}$$

(ii) $$f(0) = \arcsin\tfrac{3}{5}$$

$$f'(0) = \frac{1}{\sqrt{1 - (\tfrac{3}{5})^2}} = \frac{1}{\sqrt{1 - \tfrac{9}{25}}} = \frac{1}{\sqrt{\tfrac{16}{25}}} = \frac{5}{4}$$

$$f''(x) = \tfrac{3}{5}\left(1 - (\tfrac{3}{5})^2\right)^{-\frac{3}{2}} = \tfrac{3}{5}(1 - \tfrac{9}{25})^{-\frac{3}{2}} = \tfrac{3}{5}(\tfrac{16}{25})^{-\frac{3}{2}} = \tfrac{3}{5} \times \tfrac{125}{64} = \tfrac{75}{64}$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

$$= \arcsin\tfrac{3}{5} + \tfrac{5}{4}x + \tfrac{1}{2}x^2 \times \tfrac{75}{64} + \ldots$$

$$= \arcsin\tfrac{3}{5} + \tfrac{5}{4}x + \tfrac{75}{128}x^2 + \ldots$$

$p = \tfrac{4}{5}, \quad q = \tfrac{75}{128}$

(iii) $$\int_0^{0.1} f(x)\,dx \approx \int_0^{0.1} \left(\arcsin\tfrac{3}{5} + \tfrac{5}{4}x + \tfrac{75}{128}x^2\right)dx$$

$$= \left[x\arcsin\tfrac{3}{5} + \tfrac{5}{8}x^2 + \tfrac{25}{128}x^3\right]_0^{0.1}$$

$$= 0.1\arcsin\tfrac{3}{5} + \tfrac{5}{8} \times 0.1^2 + \tfrac{25}{128} \times 0.1^3$$

$$= 0.0708 \text{ (3 d.p.)}$$