Section 1: The method of differences

Solutions to Exercise level 2

1. (i) $\dfrac{1}{(r+1)(r+2)(r+3)} = \dfrac{A}{r+1} + \dfrac{B}{r+2} + \dfrac{C}{r+3}$

$1 = A(r+2)(r+3) + B(r+1)(r+3) + C(r+1)(r+2)$

Putting $r = -1 \Rightarrow 1 = 1 \times 2A \Rightarrow A = \frac{1}{2}$

Putting $r = -2 \Rightarrow 1 = -1 \times 1B \Rightarrow B = -1$

Putting $r = -3 \Rightarrow 1 = -2 \times -1C \Rightarrow C = \frac{1}{2}$

$$\frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{2(r+1)} - \frac{1}{r+2} + \frac{1}{2(r+3)}$$

(ii)
$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)(r+3)} = \sum_{r=1}^{n} \left(\frac{1}{2(r+1)} - \frac{2}{2(r+2)} + \frac{1}{2(r+3)} \right)$$
$$= \left(\frac{1}{4} - \frac{2}{6} + \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{2}{8} + \frac{1}{10}\right) + \left(\frac{1}{8} - \frac{2}{10} + \frac{1}{12}\right) +$$
$$\ldots + \left(\frac{1}{2n} - \frac{2}{2(n+1)} + \frac{1}{2(n+2)}\right)$$
$$+ \left(\frac{1}{2(n+1)} - \frac{2}{2(n+2)} + \frac{1}{2(n+3)}\right)$$
$$= \frac{1}{4} - \frac{2}{6} + \frac{1}{6} + \frac{1}{2(n+2)} - \frac{2}{2(n+2)} + \frac{1}{2(n+3)}$$
$$= \frac{1}{12} - \frac{1}{2(n+2)} + \frac{1}{2(n+3)}$$

(iii) As $n \to \infty$, $\dfrac{1}{2(n+2)} \to 0$ and $\dfrac{1}{2(n+3)} \to 0$

so $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(r+1)(r+2)(r+3)} = \frac{1}{12}$

2.
$$r(r+1)(r+2) - (r-1)r(r+1) = r(r+1)(r+2-(r-1))$$
$$= r(r+1)(r+2-r+1)$$
$$= 3r(r+1)$$

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$$\sum_{r=1}^{n} 3r(r+1) = \sum_{r=1}^{n}\big(r(r+1)(r+2)-(r-1)r(r+1)\big)$$
$$= \cancel{(1\times2\times3)} - (0\times1\times2) + \cancel{(2\times3\times4)} - \cancel{(1\times2\times3)}$$
$$+\ldots+n(n+1)(n+2) - \cancel{(n-1)n(n+1)}$$
$$= n(n+1)(n+2)$$
$$\sum_{r=1}^{n} r(r+1) = \tfrac{1}{3}n(n+1)(n+2)$$

3. $$r(r+1)(r+2)(r+3)-(r-1)r(r+1)(r+2) = r(r+1)(r+2)\big(r+3-(r-1)\big)$$
$$= 4r(r+1)(r+2)$$
$$\sum_{r=1}^{n} 4r(r+1)(r+2) = \sum_{r=1}^{n}\big(r(r+1)(r+2)(r+3)-(r-1)r(r+1)(r+2)\big)$$
$$= \cancel{(1\times2\times3\times4)} - (0\times1\times2\times3) + \cancel{(2\times3\times4\times5)} - \cancel{(1\times2\times3\times4)}$$
$$+\ldots+n(n+1)(n+2)(n+3) - \cancel{(n-1)n(n+1)(n+2)}$$
$$= n(n+1)(n+2)(n+3)$$
$$\sum_{r=1}^{n} r(r+1)(r+2) = \tfrac{1}{4}n(n+1)(n+2)(n+3)$$

4. (i) $$\frac{4}{(2r-1)(2r+1)(2r+3)} = \frac{A}{2r-1} + \frac{B}{2r+1} + \frac{C}{2r+3}$$
$$4 = A(2r+1)(2r+3) + B(2r-1)(2r+3) + C(2r-1)(2r+1)$$
Putting $r = \tfrac{1}{2} \quad \Rightarrow 4 = 2\times4A \quad \Rightarrow A = \tfrac{1}{2}$

Putting $r = -\tfrac{1}{2} \quad \Rightarrow 4 = -2\times2B \quad \Rightarrow B = -1$

Putting $r = -\tfrac{3}{2} \quad \Rightarrow 4 = -4\times-2C \quad \Rightarrow C = \tfrac{1}{2}$

$$\frac{4}{(2r-1)(2r+1)(2r+3)} = \frac{1}{2(2r-1)} - \frac{1}{2r+1} + \frac{1}{2(2r+3)}$$

(ii) $$\sum_{r=1}^{n}\frac{4}{(2r-1)(2r+1)(2r+3)} = \sum_{r=1}^{n}\left(\frac{1}{2(2r-1)} - \frac{2}{2(2r+1)} + \frac{1}{2(2r+3)}\right)$$

$$= \left(\frac{1}{2} - \frac{2}{6} + \frac{1}{10}\right) + \left(\frac{1}{6} - \frac{2}{10} + \frac{1}{14}\right) + \left(\frac{1}{10} - \frac{2}{14} + \frac{1}{18}\right)$$

$$+ \ldots + \left(\frac{1}{2(2n-3)} - \frac{2}{2(2n-1)} + \frac{1}{2(2n+1)}\right)$$

$$+ \left(\frac{1}{2(2n-1)} - \frac{2}{2(2n+1)} + \frac{1}{2(2n+3)}\right)$$

$$= \frac{1}{2} - \frac{2}{6} + \frac{1}{6} + \frac{1}{2(2n+1)} - \frac{2}{2(2n+1)} + \frac{1}{2(2n+3)}$$

$$= \frac{1}{3} - \frac{1}{2(2n+1)} + \frac{1}{2(2n+3)}$$