Section 2: Applications of de Moivre's theorem

Solutions to Exercise level 3

1. $\cos 5x = \mathrm{Re}(\cos x + i\sin x)^5$

$$\begin{aligned}
&= \cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x \\
&= \cos^5 x - 10\cos^3 x(1-\cos^2 x) + 5\cos x(1-\cos^2 x)^2 \\
&= \cos^5 x - 10\cos^3 x + 10\cos^5 x + 5\cos x - 10\cos^3 x + 5\cos^5 x \\
&= 16\cos^5 x - 20\cos^3 x + 5\cos x
\end{aligned}$$

$\cos 3x = \mathrm{Re}(\cos x + i\sin x)^3$

$$\begin{aligned}
&= \cos^3 x - 3\cos x\sin^2 x \\
&= \cos^3 x - 3\cos x(1-\cos^2 x) \\
&= \cos^3 x - 3\cos x + 3\cos^3 x \\
&= 4\cos^3 x - 3\cos x
\end{aligned}$$

$$p\left(16\cos^5 x - 20\cos^3 x + 5\cos x\right) + q\left(4\cos^3 x - 3\cos x\right) + r\cos\theta$$
$$\equiv \cos^5\theta + \cos^3\theta + \cos\theta$$

$$16p\cos^5 x + (-20p + 4q)\cos^3 x + (5p - 3q + r)\cos x$$
$$\equiv \cos^5\theta + \cos^3\theta + \cos\theta$$

Comparing coefficients: $16p = 1 \quad \Rightarrow p = \frac{1}{16}$

$$-20p + 4q = 1 \quad \Rightarrow q = \frac{1+20p}{4} = \frac{9}{16}$$

$$5p - 3q + r = 1 \quad \Rightarrow r = 1 - 5p + 3q = \frac{19}{8}$$

2. $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta - i\sin n\theta$

Adding: $z^n + z^{-n} = 2\cos n\theta \quad \Rightarrow \cos n\theta = \dfrac{z^n + z^{-n}}{2}$

Subtracting: $z^n - z^{-n} = 2i\sin n\theta \quad \Rightarrow \sin n\theta = \dfrac{z^n - z^{-n}}{2i}$

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$$\cos n\theta \times \sin m\theta = \left(\frac{z^n + z^{-n}}{2}\right)\left(\frac{z^m - z^{-m}}{2i}\right)$$
$$= \frac{z^{n+m} + z^{n-m} - z^{-n+m} - z^{-m-n}}{4i}$$
$$= \frac{1}{2}\left(\frac{z^{n+m} - z^{-(n+m)}}{2i}\right) + \frac{1}{2}\left(\frac{z^{n-m} - z^{-(n-m)}}{2i}\right)$$
$$= \tfrac{1}{2}\sin(n+m)\theta + \tfrac{1}{2}\sin(m-n)\theta$$

3. Let $C = \sum_{k=0}^{n} {}^nC_r \cos(a+bk)$.

$$= \cos a + {}^nC_1\cos(a+b) + {}^nC_2\cos(a+2b) + \ldots + {}^nC_n\cos(a+nb)$$

Let $S = \sum_{k=0}^{n} {}^nC_r \sin(a+bk)$

$$= \sin a + {}^nC_1\sin(a+b) + {}^nC_2\sin(a+2b) + \ldots + {}^nC_n\sin(a+nb)$$

$$C + iS = \cos a + i\sin a + {}^nC_1\left(\cos(a+b) + i\sin(a+b)\right) + {}^nC_2\left(\cos(a+2b) + i\sin(a+2b)\right) +$$
$$\ldots + {}^nC_n\left(\cos(a+nb) + i\sin(a+nb)\right)$$
$$= e^{ia} + {}^nC_1 e^{i(a+b)} + {}^nC_2 e^{i(a+2b)} + \ldots + {}^nC_n e^{i(a+nb)}$$
$$= e^{ia}\left(1 + {}^nC_1 e^{bi} + {}^nC_2 e^{2bi} + \ldots + {}^nC_n e^{nbi}\right)$$
$$= e^{ia}\left(1 + e^{bi}\right)^n$$

$$1 + e^{bi} = 1 + \cos b + i\sin b$$
$$= 1 + \cos\frac{2b}{2} + i\sin\frac{2b}{2}$$
$$= 2\cos^2\frac{b}{2} + 2i\sin\frac{b}{2}\cos\frac{b}{2}$$
$$= 2\cos\frac{b}{2}\left(\cos\frac{b}{2} + i\sin\frac{b}{2}\right)$$
$$= 2\cos\frac{b}{2}e^{\frac{1}{2}bi}$$

$$c + iS = e^{ia}\left(2e^{\frac{1}{2}bi}\cos\tfrac{1}{2}b\right)^n$$
$$= e^{ia} \times 2^n e^{\frac{1}{2}nbi}\cos^n\tfrac{1}{2}b$$
$$= 2^n e^{(a+\frac{1}{2}nb)i}\cos^n\tfrac{1}{2}b$$
$$= 2^n\left(\cos(a+\tfrac{1}{2}nb)\theta + i\sin(a+\tfrac{1}{2}nb)\theta\right)\cos^n\tfrac{1}{2}b$$

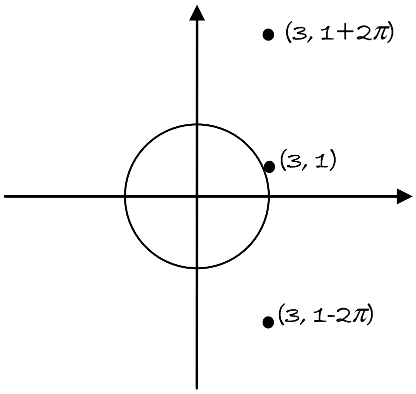
So $C = 2^n\cos(a+\tfrac{1}{2}nb)\theta\cos^n\tfrac{1}{2}b$

4. Let $z = x + iy$

$$e^{x+iy} = 1 - i \quad \Rightarrow e^x e^{iy} = \sqrt{2}e^{-\frac{1}{4}\pi i}$$

But since $e^{2n\pi i} = 1$ for all integer values of n, can write $e^x e^{i(y+2n\pi)} = \sqrt{2}e^{-\frac{1}{4}\pi i}$
Hence $e^x = \sqrt{2} \Rightarrow x = \ln\sqrt{2} = \frac{1}{2}\ln 2$
and $y + 2n\pi = -\frac{1}{4}\pi \Rightarrow y = -(2n+\frac{1}{4})\pi$
So $z = \frac{1}{2}\ln 2 - (2n+\frac{1}{4})\pi i$

5. (i) $e^{3+i} = e^{3+(1\pm 2\pi)i} = e^{3+(1\pm 4\pi)i} = \ldots$
so solutions are $z = 3+i, 3+(1\pm 2\pi)i, (3\pm 4\pi)i, \ldots$



(iii) An algebraic number can be written as the solution to a polynomial equation with integer coefficients, e.g. any fraction can be written as the solution to $ax + b = 0$, $\sqrt{2}$ can be written as the solution to $x^2 = 2$, and i can be written as the solution to $x^2 + 1 = 0$.
A transcendental number is one that is not algebraic. Examples include π and e.

(iv) If $p + i(q + 2n\pi)$ is on the circle centre O and radius k, where k is a whole number, then $p^2 + (q + 2n\pi)^2 = k^2$. This means that unless $n = 0$, π is a solution to the integer polynomial equation $p^2 + (q + 2nx)^2 = k^2$, which is not possible since π is a transcendental number. So n must be 0, and hence Im(z) = q which is an integer.

6. (i) $z = x + iy \Rightarrow \text{LHS} = e^{x+iy} = e^x e^{iy}$

(ii) $z = re^{i\theta} \Rightarrow \text{RHS} = \left(re^{i\theta}\right)^e = r^e e^{i\theta e}$

(iii) $e^x e^{iy} = r^e e^{i\theta e}$

$y + 2n\pi = e\theta \Rightarrow r\sin\theta + 2n\pi = e\theta$

$$\Rightarrow r = \frac{e\theta - 2n\pi}{\sin\theta}$$

$e^x = r^e \Rightarrow e^{r\cos\theta} = r^e$

$$\Rightarrow e^{(e\theta-2n\pi)\cot\theta}-\left(\frac{e\theta-2n\pi}{\sin\theta}\right)^e=0$$

(iv) Graphing $y=e^{(e\theta-2\pi)\cot\theta}-\left(\frac{e\theta-2\pi}{\sin\theta}\right)^e$ gives

$$\theta=2.48870\Rightarrow r=0.79311$$

$$\Rightarrow x=-0.62999,\ y=0.48180$$

Check: $e^z=e^{-0.62999}(\cos(0.48180)+i\sin(0.48180))$

$$=0.472+0.247i$$

$$z^e=0.79311^e(\cos(2.48870e)+i\sin(2.48870e))$$

$$=0.472+2.47i$$

7. (i) $1-e^{\frac{2n\pi}{3}i}=1-\cos\frac{2n\pi}{3}-i\sin\frac{2n\pi}{3}$

$$=2\sin^2\frac{n\pi}{3}-2i\sin\frac{n\pi}{3}\cos\frac{n\pi}{3}$$

$$=2\sin\frac{n\pi}{3}\left(\sin\frac{n\pi}{3}-i\cos\frac{n\pi}{3}\right)$$

$$=-2i\sin\frac{n\pi}{3}\left(i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}\right)=-2ie^{\frac{n\pi}{3}i}\sin\frac{n\pi}{3}$$

$1+e^{\frac{2n\pi}{3}i}=1+\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$

$$=2\cos^2\frac{n\pi}{3}+2i\sin\frac{n\pi}{3}\cos\frac{n\pi}{3}$$

$$=2\cos\frac{n\pi}{3}\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)=2e^{\frac{n\pi}{3}i}\sin\frac{n\pi}{3}$$

(ii) $(z+1)^3=(z-1)^3$

$$z^3+3z^2+3z+1=z^3-3z^2+3z-1$$

$$6z^2+2=0$$

$$z=\pm\frac{1}{\sqrt{3}}i$$

(iii) The cube roots of 1 are 1, $e^{\frac{2\pi i}{3}}$ and $e^{-\frac{2\pi i}{3}}$

$$\left(\frac{z+1}{z-1}\right)^3=1\ \Rightarrow\frac{z+1}{z-1}=\sqrt[3]{1}$$

$$\Rightarrow z+1=\sqrt[3]{1}(z-1)$$

$$\Rightarrow z(\sqrt[3]{1}-1)=1+\sqrt[3]{1}$$

$$\Rightarrow z=\frac{\sqrt[3]{1}+1}{\sqrt[3]{1}-1}$$

For $\sqrt[3]{1} = 1$, z is undefined

For $\sqrt[3]{1} = e^{\frac{2\pi i}{3}}$, $z = \dfrac{e^{\frac{2\pi i}{3}} + 1}{e^{\frac{2\pi i}{3}} - 1} = \dfrac{-2ie^{\frac{n\pi}{3}i} \sin\frac{n\pi}{3}}{2e^{\frac{n\pi}{3}i} \cos\frac{n\pi}{3}} = -i\tan\dfrac{n\pi}{3}$

$$= -\frac{1}{\sqrt{3}}i$$

For $\sqrt[3]{1} = e^{-\frac{2\pi i}{3}}$, $z = \dfrac{e^{-\frac{2\pi i}{3}} + 1}{e^{-\frac{2\pi i}{3}} - 1} = \dfrac{-2ie^{-\frac{n\pi}{3}i} \sin\left(-\frac{n\pi}{3}\right)}{2e^{-\frac{n\pi}{3}i} \cos\left(-\frac{n\pi}{3}\right)} = -i\tan\left(-\dfrac{n\pi}{3}\right)$

$$= \frac{1}{\sqrt{3}}i$$

Both methods give the same answer.

8. (i) $1 + e^{i\theta} = 1 + \cos\theta + i\sin\theta$

$$= 2\cos^2\tfrac{\theta}{2} + 2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$= 2\cos\tfrac{\theta}{2}(\cos\tfrac{\theta}{2} + i\sin\tfrac{\theta}{2})$$
$$= 2\cos\tfrac{\theta}{2}e^{\frac{i\theta}{2}}$$

$-1 + e^{i\theta} = -1 + \cos\theta + i\sin\theta$

$$= -2\sin^2\tfrac{\theta}{2} + 2i\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$= 2\sin\tfrac{\theta}{2}(-\sin\tfrac{\theta}{2} + i\cos\tfrac{\theta}{2})$$
$$= 2i\sin\tfrac{\theta}{2}(i\sin\tfrac{\theta}{2} + \cos\tfrac{\theta}{2})$$
$$= 2i\sin\tfrac{\theta}{2}e^{\frac{i\theta}{2}}$$

(ii) $z_1 = 1 + e^{i\theta}$, $z_2 = e^{i\theta}$

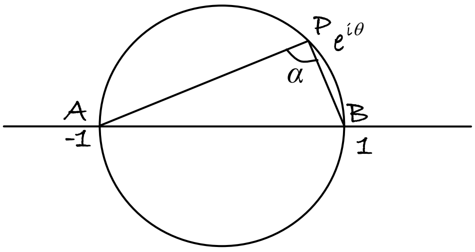
The diagonals of the rhombus are represented by the complex numbers z_1 and $z_2 - 1$.

Multiplying by the complex number $e^{i\alpha}$ has the effect of rotating anticlockwise through an angle of α, so

$$z_1 \times e^{i\alpha} = k(z_2 - 1)$$
$$(1 + e^{i\theta})e^{j\alpha} = k(e^{i\theta} - 1)$$
$$2\cos\tfrac{\theta}{2}e^{\frac{i\theta}{2}}e^{i\alpha} = k \times 2i\sin\tfrac{\theta}{2}e^{\frac{j\theta}{2}}$$
$$\cos\tfrac{\theta}{2}e^{i\alpha} = ki\sin\tfrac{\theta}{2}$$
$$e^{i\alpha} = ki\tan\tfrac{\theta}{2}$$

So $\text{Re}(e^{i\alpha}) = 0$ and therefore $\cos\alpha = 0 \Rightarrow \alpha = 90°$

(iii)



PA is represented by the complex number $-1-e^{i\theta}$

PB is represented by the complex number $1-e^{i\theta}$

Multiplying by $e^{i\alpha}$ rotates through α, so

$$(-1-e^{i\theta})ke^{i\alpha} = 1-e^{i\theta}$$

so $e^{i\alpha} = -\frac{1}{k}\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)$

$$= -\frac{1}{k}\left(\frac{2i\sin\frac{\theta}{2}e^{\frac{i\theta}{2}}}{2\cos\frac{\theta}{2}e^{\frac{i\theta}{2}}}\right)$$

$$= -\frac{1}{k}i\tan\tfrac{\theta}{2}$$

So $\text{Re}(e^{i\alpha}) = 0$ and therefore $\cos\alpha = 0 \Rightarrow \alpha = 90°$.