# Section 2: Applications of de Moivre's theorem

## Solutions to Exercise level 2

1. (i) $(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos(3\theta) + i\sin(3\theta)$$

Equating real parts:
$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Equating imaginary parts:
$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

(ii) $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta} = \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

Dividing through by $\cos^3\theta$:
$$\begin{aligned}\tan 3\theta &= \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta} \\ &= \frac{3\tan\theta(1+\tan^2\theta) - 4\tan^3\theta}{4 - 3(1+\tan^2\theta)} \\ &= \frac{3\tan\theta + 3\tan^3\theta - 4\tan^3\theta}{4 - 3 - 3\tan^2\theta} \\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

2. $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$$z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

Adding: $z^n + z^{-n} = 2\cos n\theta \quad \Rightarrow \cos n\theta = \dfrac{z^n + z^{-n}}{2}$

Subtracting: $z^n - z^{-n} = 2i\sin n\theta \quad \Rightarrow \sin n\theta = \dfrac{z^n - z^{-n}}{2i}$


11/06/18 © MEI

# Edexcel FM Complex numbers 2 Exercise solutions

$$
\begin{aligned}
2\cos(n\theta)\sin(n\theta) &= 2\left(\frac{z^n + z^{-n}}{2}\right)\left(\frac{z^n - z^{-n}}{2i}\right)\\
&= \frac{z^{2n} + 1 - 1 - z^{-2n}}{2i}\\
&= \frac{z^{2n} - z^{-2n}}{2i}\\
&= \sin(2n\theta)
\end{aligned}
$$

3.
$$
\begin{aligned}
\sin 5\theta &= \text{Im}(\cos 5\theta + i\sin 5\theta)\\
&= \text{Im}(\cos\theta + i\sin\theta)^5\\
&= \text{Im}\big(\cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2\\
&\qquad + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5\big)\\
&= \text{Im}\big(\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta\\
&\qquad - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta\big)\\
&= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\\
&= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta\\
&= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta\\
&= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta
\end{aligned}
$$

Only the imaginary part is required

4. If $z = \cos\theta + i\sin\theta$,

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$$
\begin{aligned}
\left(z+\frac{1}{z}\right)^5\left(z-\frac{1}{z}\right)^2 &= \left(z^5 + \frac{5z^4}{z} + \frac{10z^3}{z^2} + \frac{10z^2}{z^3} + \frac{5z}{z^4} + \frac{1}{z^5}\right)\left(z^2 - 2 + \frac{1}{z^2}\right)\\
&= \left(z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}\right)\left(z^2 - 2 + \frac{1}{z^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= z^7 + 5z^5 + 10z^3 + 10z + \frac{5}{z} + \frac{1}{z^3} - 2z^5 - 10z^3 - 20z - \frac{20}{z} - \frac{10}{z^3} - \frac{2}{z^5}\\
&\qquad + z^3 + 5z + \frac{10}{z} + \frac{10}{z^3} + \frac{5}{z^5} + \frac{1}{z^7}
\end{aligned}
$$

$$(2\cos\theta)^5(2i\sin\theta)^2 = z^7 + \frac{1}{z^7} + 3z^5 + \frac{3}{z^5} + z^3 + \frac{1}{z^3} - 5z - \frac{5}{z}$$

$$32\cos^5\theta \times -4\sin^2\theta = 2\cos 7\theta + 3\times 2\cos 5\theta + 2\cos 3\theta - 5\times 2\cos\theta$$

$$-128\cos^5\theta\sin^2\theta = 2\cos 7\theta + 6\cos 5\theta + 2\cos 3\theta - 10\cos\theta$$

$$\cos^5\theta\sin^2\theta = \tfrac{1}{64}(5\cos\theta - \cos 3\theta - 3\cos 5\theta - \cos 7\theta)$$


11/06/18 © MEI

# Edexcel FM Complex numbers 2 Exercise solutions

5. (i) $e^{ik\theta} = \cos k\theta + i\sin k\theta$

$$e^{-ik\theta} = \cos(-k\theta) + i\sin(-k\theta)$$
$$= \cos k\theta - i\sin k\theta$$

$$e^{2i\theta} - 1 = e^{i\theta}(e^{i\theta} - e^{-i\theta})$$
$$= e^{i\theta}(\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta))$$
$$= e^{i\theta}(2i\sin\theta)$$
$$= 2ie^{i\theta}\sin\theta$$

(ii) $C + iS = \cos\theta + i\sin\theta + \cos 3\theta + i\sin 3\theta + \cos 5\theta + i\sin 5\theta +$

$$\ldots + \cos(2n-1)\theta + i\sin(2n-1)\theta$$
$$= e^{i\theta} + e^{3i\theta} + e^{5i\theta} + \ldots + e^{(2n-1)i\theta}$$

This is a geometric series with $n$ terms, first term $e^{i\theta}$, common ratio $e^{2i\theta}$.

$$\text{Sum} = \frac{e^{i\theta}(e^{2in\theta} - 1)}{e^{2i\theta} - 1}$$

(iii) $C + iS = \dfrac{e^{i\theta}(e^{2in\theta} - 1)}{e^{2i\theta} - 1} = \dfrac{e^{i\theta} \times 2je^{ni\theta}\sin n\theta}{2ie^{j\theta}\sin\theta} = \dfrac{\sin n\theta}{\sin\theta}e^{ni\theta}$

This is of the form $re^{i\theta}$, so $|C + iS| = \dfrac{\sin n\theta}{\sin\theta}$

$$\arg(C + iS) = n\theta$$

(iv) $C + iS = \dfrac{\sin n\theta}{\sin\theta}(\cos n\theta + i\sin n\theta)$

Equating real parts: $C = \dfrac{\sin n\theta\cos n\theta}{\sin\theta}$

Equating imaginary parts: $S = \dfrac{\sin^2 n\theta}{\sin\theta}$

6. (i) $(1 - ke^{i\theta})(1 - ke^{-i\theta}) = 1 - k(e^{i\theta} + e^{-i\theta}) + k^2$

$$= 1 - k(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta) + k^2$$
$$= 1 - 2k\cos\theta + k^2$$

(ii) $C + iS = k(\cos\theta + i\sin\theta) + k^2(\cos 2\theta + i\sin 2\theta)$

$$+ k^3(\cos 3\theta + i\sin 3\theta)\ldots$$
$$= ke^{i\theta} + k^2e^{2i\theta} + k^3e^{3i\theta} + \ldots$$

This is an infinite geometric series with first term $ke^{i\theta}$ and common ratio $ke^{i\theta}$.


11/06/18 © MEI

# Edexcel FM Complex numbers 2 Exercise solutions

(iii) $$C + iS = \frac{ke^{i\theta}}{1 - ke^{i\theta}}$$
$$= \frac{ke^{i\theta}(1 - ke^{-i\theta})}{(1 - ke^{i\theta})(1 - ke^{-i\theta})}$$
$$= \frac{ke^{i\theta} - k^2}{1 - 2k\cos\theta + k^2}$$
$$= \frac{k(\cos\theta + i\sin\theta) - k^2}{1 - 2k\cos\theta + k^2}$$

Equating real parts: $C = \dfrac{k\cos\theta - k^2}{1 - 2k\cos\theta + k^2}$

Equating imaginary parts: $S = \dfrac{k\sin\theta}{1 - 2k\cos\theta + k^2}$

(iv) $C = 0 \quad \Rightarrow k\cos\theta - k^2 = 0$
$$\Rightarrow k(\cos\theta - k) = 0$$
$$\Rightarrow \cos\theta = k$$
$$S = \frac{\cos\theta\sin\theta}{1 - 2\cos\theta\cos\theta + \cos^2\theta} = \frac{\sin\theta\cos\theta}{1 - \cos^2\theta} = \frac{\sin\theta\cos\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

7. (i) (a) $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$
$$= 2\cos\theta$$
(b) $(1 - 3e^{i\theta})(1 - 3e^{-i\theta}) = 1 - 3(e^{i\theta} + e^{-i\theta}) + 9$
$$= 10 - 3(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$$
$$= 10 - 6\cos\theta$$

(ii) $C + iS = \cos\theta + i\sin\theta + 3(\cos 2\theta + i\sin 2\theta) + 9(\cos 3\theta + i\sin 3\theta)$
$$+ \ldots + 3^{n-1}(\cos n\theta + i\sin n\theta)$$
$$= e^{i\theta} + 3e^{2i\theta} + 3^2 e^{3i\theta} + \ldots + 3^{n-1}e^{ni\theta}$$

This is a geometric series with $n$ terms, with first term $e^{i\theta}$ and common ratio $3e^{i\theta}$.

$$C + iS = \frac{e^{i\theta}(1 - 3^n e^{ni\theta})}{1 - 3e^{i\theta}}$$
$$= \frac{e^{i\theta}(1 - 3^n e^{ni\theta})(1 - 3e^{-i\theta})}{(1 - 3e^{i\theta})(1 - 3e^{-i\theta})}$$
$$= \frac{e^{i\theta} - 3^n e^{(n+1)i\theta} - 3 + 3^{n+1}e^{ni\theta}}{10 - 6\cos\theta}$$

Real parts: $C = \dfrac{\cos\theta - 3^n\cos(n+1)\theta - 3 + 3^{n+1}\cos n\theta}{10 - 6\cos\theta}$

Imaginary parts: $S = \dfrac{\sin\theta - 3^n\sin(n+1)\theta + 3^{n+1}\sin n\theta}{10 - 6\cos\theta}$


11/06/18 © MEI

8. (i) $\left(1-\frac{e^{i\theta}}{2}\right)\left(1-\frac{e^{-i\theta}}{2}\right)=1-\frac{e^{i\theta}}{2}-\frac{e^{-i\theta}}{2}+\frac{1}{4}$

$$=\frac{5}{4}-\tfrac{1}{2}(\cos\theta+i\sin\theta)-\tfrac{1}{2}(\cos\theta-i\sin\theta)$$

$$=\frac{5}{4}-\cos\theta$$

(ii) $C+iS=\cos\theta+i\sin\theta+\frac{\cos 2\theta}{2}+i\frac{\sin 2\theta}{2}+\frac{\cos 3\theta}{4}+i\frac{\sin 3\theta}{4}+\ldots$

$$=e^{i\theta}+\tfrac{1}{2}e^{2i\theta}+\tfrac{1}{4}e^{3i\theta}+\ldots$$

This is a geometric series with $a=e^{i\theta}$ and $r=\tfrac{1}{2}e^{i\theta}$

$$C+iS=\frac{e^{i\theta}}{1-\tfrac{1}{2}e^{i\theta}}$$

$$=\frac{e^{i\theta}(1-\tfrac{1}{2}e^{-i\theta})}{(1-\tfrac{1}{2}e^{i\theta})(1-\tfrac{1}{2}e^{-i\theta})}$$

$$=\frac{e^{i\theta}-\tfrac{1}{2}}{\tfrac{5}{4}-\cos\theta}\quad\text{(from part (i))}$$

$$=\frac{4(\cos\theta+i\sin\theta)-2}{5-4\cos\theta}$$

Equating real parts: $C=\frac{4\cos\theta-2}{5-4\cos\theta}$

Equating imaginary parts: $S=\frac{4\sin\theta}{5-4\cos\theta}$

9. (i) $2\cos\left(\frac{3\theta}{2}\right)e^{\frac{3}{2}i\theta}=2\cos\frac{3\theta}{2}\left(\cos\frac{3\theta}{2}+i\sin\frac{3\theta}{2}\right)$

$$=2\cos^2\frac{3\theta}{2}+2i\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}$$

$$=\cos 3\theta+1+i\sin 3\theta$$

$$=1+e^{3i\theta}$$

(ii) $C=1+\binom{n}{1}\cos 3\theta+\binom{n}{2}\cos 6\theta+\ldots+\binom{n}{n}\cos 3n\theta$

$$S=1+\binom{n}{1}\sin 3\theta+\binom{n}{2}\sin 6\theta+\ldots+\binom{n}{n}\sin 3n\theta$$


11/06/18 © MEI

$$C + iS = 1 + i + \binom{n}{1}(\cos 3\theta + i\sin 3\theta) + \binom{n}{2}(\cos 6\theta + i\sin 6\theta) +$$

$$\ldots + \binom{n}{n}(\cos 3n\theta + i\sin 3n\theta)$$

$$= 1 + i + \binom{n}{1}e^{3i\theta} + \binom{n}{2}e^{6i\theta} + \ldots + \binom{n}{n}e^{3ni\theta}$$

Using the binomial theorem:

$$C + iS = i + \left(1 + e^{3i\theta}\right)^n$$

$$= i + \left(2\cos\left(\frac{3\theta}{2}\right)e^{\frac{3}{2}i\theta}\right)^n \quad \text{(from part (i))}$$

$$= i + 2^n \cos^n\left(\frac{3\theta}{2}\right)e^{\frac{3}{2}ni\theta}$$

$$= i + 2^n \cos^n\left(\frac{3\theta}{2}\right)\left(\cos\left(\frac{3n\theta}{2}\right) + i\sin\left(\frac{3n\theta}{2}\right)\right)$$

Equating real parts: $C = 2^n \cos^n\left(\frac{3\theta}{2}\right)\cos\left(\frac{3n\theta}{2}\right)$

Equating imaginary parts: $S = 1 + 2^n \cos^n\left(\frac{3\theta}{2}\right)\sin\left(\frac{3n\theta}{2}\right)$

10\. (i) $r = \sqrt{1+3} = 2$

$\theta = \arctan\frac{\sqrt{3}}{1} = \frac{\pi}{3}$

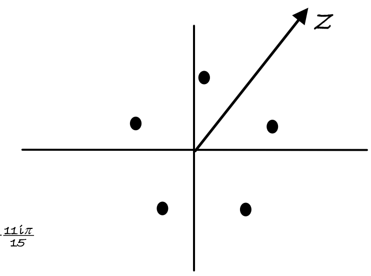
$z = 2e^{\frac{i\pi}{3}}$

(ii) Modulus of 5th roots is $2^{\frac{1}{5}}$

Arguments of 5th roots are $\frac{\pi}{15} \pm \frac{2n\pi}{5}$

so arguments are $\frac{\pi}{15}, \frac{7\pi}{15}, \frac{13\pi}{15}, -\frac{5\pi}{15}, -\frac{11\pi}{15}$

So 5th roots are $2^{\frac{1}{5}}e^{\frac{i\pi}{15}}, 2^{\frac{1}{5}}e^{\frac{7i\pi}{15}}, 2^{\frac{1}{5}}e^{\frac{13i\pi}{15}}, 2^{\frac{1}{5}}e^{-\frac{i\pi}{3}}, 2^{\frac{1}{5}}e^{-\frac{11i\pi}{15}}$




(iii) $z_1 = 2^{\frac{1}{5}}e^{\frac{i\pi}{15}}, z_2 = 2^{\frac{1}{5}}e^{\frac{7i\pi}{15}}$

$$w = \tfrac{1}{2}\left(2^{\frac{1}{5}}e^{\frac{i\pi}{15}} + 2^{\frac{1}{5}}e^{\frac{7i\pi}{15}}\right)$$

$$= \tfrac{1}{2}2^{\frac{1}{5}}e^{\frac{i\pi}{15}}\left(1 + e^{\frac{6i\pi}{15}}\right)$$

$$= \tfrac{1}{2}2^{\frac{1}{5}}e^{\frac{i\pi}{15}}\left(1 + e^{\frac{2i\pi}{5}}\right)$$

Using $1 + e^{2i\theta} = 2e^{i\theta}\cos\theta$ with $\theta = \frac{\pi}{5}$



11/06/18 © MEI

# Edexcel FM Complex numbers 2 Exercise solutions

$w = \frac{1}{2} 2^{\frac{1}{5}} e^{\frac{i\pi}{15}} \times 2e^{\frac{i\pi}{5}} \cos\frac{\pi}{5}$

$= 2^{\frac{1}{5}} e^{\frac{4i\pi}{15}} \cos\frac{\pi}{5}$

$= 2^{\frac{1}{5}} \cos\frac{\pi}{5}\left(\cos\frac{4\pi}{15} + i\sin\frac{4\pi}{15}\right)$

$w^n = 2^{\frac{n}{5}} \cos^n\frac{\pi}{5}\left(\cos\frac{4n\pi}{15} + i\sin\frac{4n\pi}{15}\right)$

For $w^n$ to be real, $\dfrac{4n}{15}$ must be an integer, so smallest value of $n = 15$


11/06/18 © MEI