Section 2: Applications of de Moivre's theorem

Solutions to Exercise level 1

1. $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta)$

   $\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos(2\theta) + i\sin(2\theta)$

   Equating real parts gives $\cos 2\theta = \cos^2\theta - \sin^2\theta$

   Equating imaginary parts gives $\sin 2\theta = 2\sin\theta\cos\theta$

2. (i) The modulus of 2 + 2i is $\sqrt{2^2+2^2} = 2\sqrt{2}$

   The argument of 2 + 2i is $\tan^{-1}(1) = \frac{\pi}{4}$

   $2 + 2i = 2\sqrt{2}e^{\frac{i\pi}{4}}$

   (ii) The modulus of $5 - 5\sqrt{3}i$ is $5\sqrt{1+3} = 10$

   The argument of $5 - 5\sqrt{3}i$ is $\tan^{-1}(\sqrt{3}) = -\frac{\pi}{3}$

   $5 - 5\sqrt{3}i = 10e^{-\frac{i\pi}{3}}$

3. (i) $e^{\frac{2\pi}{3}i} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

   $= -\frac{1}{2} + \frac{1}{2}i\sqrt{3}$

   (ii) $e^{2+\frac{\pi}{3}i} = e^2 e^{\frac{\pi}{3}i} = e^2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

   $= \frac{1}{2}e^2 + \frac{1}{2}e^2 i\sqrt{3}$

   (iii) $e^{-2-\frac{\pi}{3}i} = e^{-2}e^{-\frac{\pi}{3}i} = e^{-2}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

   $= \frac{1}{2}e^{-2} - \frac{1}{2}e^{-2}i\sqrt{3}$

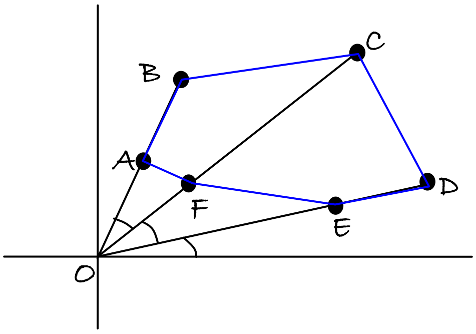
   (iv) $e^{3+2i} = e^3 e^{2i} = e^3(\cos 2 + i\sin 2)$

   $= e^3\cos 2 + e^3 i\sin 2$

Edexcel FM Complex numbers 2 Exercise solutions

4. $z = e^{\frac{1}{2}i}$ $\quad z = \frac{1}{2}e^{i}$ $\quad z = \frac{3}{2}e^{i}$

$z = e^{\frac{3}{2}i}$ $\quad z = \frac{1}{2}e^{\frac{3}{2}i}$ $\quad z = \frac{3}{2}e^{\frac{1}{2}i}$



A is $z = \frac{1}{2}e^{\frac{3}{2}i}$ $\quad$ B is $z = e^{\frac{3}{2}i}$

C is $z = \frac{3}{2}e^{i}$ $\quad$ D is $z = \frac{3}{2}e^{\frac{1}{2}i}$

E is $z = e^{\frac{1}{2}i}$ $\quad$ F is $z = \frac{1}{2}e^{i}$

All three marked angles are $\frac{1}{2}$

Area of triangle OBC $= \frac{1}{2} \times OB \times OC \times \sin BOC = \frac{1}{2} \times 1 \times \frac{3}{2} \sin \frac{1}{2} = \frac{3}{4} \sin \frac{1}{2}$

Area of triangle OCD $= \frac{1}{2} \times OC \times OD \times \sin COD = \frac{1}{2} \times \frac{3}{2} \times \frac{3}{2} \sin \frac{1}{2} = \frac{9}{8} \sin \frac{1}{2}$

Area of triangle OAF $= \frac{1}{2} \times OA \times OF \times \sin AOF = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \sin \frac{1}{2} = \frac{1}{8} \sin \frac{1}{2}$

Area of triangle OFE $= \frac{1}{2} \times OF \times OE \times \sin FOE = \frac{1}{2} \times \frac{1}{2} \times 1 \sin \frac{1}{2} = \frac{1}{4} \sin \frac{1}{2}$

Area of polygon $= \frac{3}{4} \sin \frac{1}{2} + \frac{9}{8} \sin \frac{1}{2} - \frac{1}{8} \sin \frac{1}{2} - \frac{1}{4} \sin \frac{1}{2}$

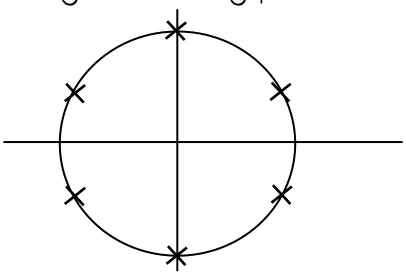
$= \frac{3}{2} \sin \frac{1}{2}$

5. $z^6 = -64 = 64e^{i\pi}$

$\therefore$ One root is $\sqrt[6]{64}\left(e^{i\pi}\right)^{\frac{1}{6}} = 2e^{\frac{i\pi}{6}}$

Hence all the 6 roots lie on a circle in the Argand diagram of radius 2.

The 6 roots are symmetrically placed on the circle with one root being $2e^{\frac{i\pi}{6}}$.



(i) The roots are $2e^{\frac{i\pi}{6}}$, $2e^{\frac{i\pi}{2}}$, $2e^{\frac{5i\pi}{6}}$, $2e^{-\frac{i\pi}{6}}$, $2e^{-\frac{i\pi}{2}}$, $2e^{-\frac{5i\pi}{6}}$.

(ii) $2e^{\frac{i\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$

$2e^{\frac{i\pi}{2}} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2i$

$2e^{\frac{5i\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$

$$2e^{-\frac{i\pi}{6}} = 2\left(\cos\tfrac{\pi}{6} - i\sin\tfrac{\pi}{6}\right) = \sqrt{3} - i$$
$$2e^{-\frac{i\pi}{2}} = 2\left(\cos\tfrac{\pi}{2} - i\sin\tfrac{\pi}{2}\right) = -2i$$
$$2e^{-\frac{5i\pi}{6}} = 2\left(\cos\tfrac{5\pi}{6} - i\sin\tfrac{5\pi}{6}\right) = -\sqrt{3} - i$$

The roots are $\sqrt{3}+i$, $2i$, $-\sqrt{3}+i$, $\sqrt{3}-i$, $-2i$, $-\sqrt{3}-i$.

6. (i) $(a+bi)^2 = 3+4i$

$a^2 + 2abi - b^2 = 3+4i$

Equating imaginary parts: $2ab = 4 \quad \Rightarrow b = \dfrac{2}{a}$

Equating real parts: $a^2 - b^2 = 3$

$$a^2 - \frac{4}{a^2} = 3$$
$$a^4 - 3a^2 - 4 = 0$$
$$(a^2-4)(a^2+1) = 0$$
$$a = \pm 2$$

$a = \pm 2 \Rightarrow b = \pm 1$

Square roots are $\pm(2+i)$

(ii) $|3+4i| = \sqrt{3^2+4^2} = 5$

$\arg(3+4i) = \arctan\frac{4}{3}$

$3+4i = 5e^{i\arctan\frac{4}{3}}$

Square roots are $\sqrt{5}e^{\frac{1}{2}i\arctan\frac{4}{3}}$ and $\sqrt{5}e^{i\left(\frac{1}{2}\arctan\frac{4}{3}-\pi\right)}$

or $\sqrt{5}e^{0.4636i}$ and $\sqrt{5}e^{-2.6779i}$

(iii) $\sqrt{5}e^{0.4636i} = \sqrt{5}\left(\cos(0.4636) + i\sin(0.4636)\right)$

$= 2+i$

$\sqrt{5}e^{0.4636i} = \sqrt{5}\left(\cos(-2.6779) + i\sin(-2.6679)\right)$

$= -2-i$