Section 1: de Moivre's theorem

Solutions to Exercise level 3

1. (i) $\cos 4x = \text{Re}(\cos x + i\sin x)^4$

$$= \cos^4 x - 6\cos^2 x\sin^2 x + \sin^4 x$$
$$= \cos^4 x - 6\cos^2 x(1-\cos^2 x) + (1-\cos^2 x)^2$$
$$= \cos^4 x - 6\cos^2 x + 6\cos^4 x + 1 - 2\cos^2 x + \cos^4 x$$
$$= 8\cos^4 x - 8\cos^2 x + 1$$

$\cos^4 x = \cos 4x$

$\cos^4 x = 8\cos^4 x - 8\cos^2 x + 1$

$7\cos^4 x - 8\cos^2 x + 1 = 0$

$(7\cos^2 x - 1)(\cos^2 x - 1) = 0$

$\cos x = \pm\dfrac{1}{\sqrt{7}}$ or ± 1

Smallest positive root is $\cos x = \dfrac{1}{\sqrt{7}} \Rightarrow x = \arccos\dfrac{1}{\sqrt{7}}$

(ii) $\sin 4x = \text{Im}(\cos x + i\sin x)^4$

$$= 4\cos^3 x\sin x - 4\cos x\sin^3 x$$

$\sin^4 x = 4\cos^3 x\sin x - 4\cos x\sin^3 x$

$\sin^4 x + 4\cos x\sin^3 x - 4\cos^3 x\sin x = 0$

$\sin x(\sin^3 x + 4\cos x\sin^2 x - 4\cos^3 x) = 0$

Since the root we are looking for is between 0 and $\frac{\pi}{2}$, we can divide through by $\sin x\cos^3 x$:

$$\tan^3 x + 4\tan^2 x - 4 = 0$$

2. (i) $z = r(\cos\theta + i\sin\theta) \Rightarrow z^7 = r^7(\cos 7\theta + i\sin 7\theta)$

For z^7 to be real, $7\theta = n\pi$ so $\theta = \dfrac{\pi}{7}$ is a possibility

$z_1 = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$ is an example.

(ii) $z = r(\cos\theta + i\sin\theta) \Rightarrow z^5 = r^5(\cos 5\theta + i\sin 5\theta)$

For z^5 to be pure imaginary, $5\theta = (2n+1)\dfrac{\pi}{2}$ so $\theta = \dfrac{\pi}{10}$ is a possibility

$z_1 = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$ is an example.

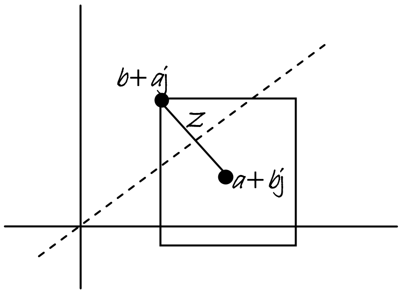
(iii) If z^7 is real and z^5 is pure imaginary, there must exist integers m and n

for which $\frac{n\pi}{7} = (2m+1)\frac{\pi}{10}$

$10n = 7(2m+1)$

LHS is even, RHS is odd, so not possible.

3.



$z = b + ai - (a + bi) = b - a - (b - a)i = (b - a)(1 - i)$

To obtain the other vertices, z needs to be rotated through 90°, which is equivalent to multiplying by i.

So the other vertices are:

$a + bi + i(b - a)(1 - i) = b + (2b - a)i$

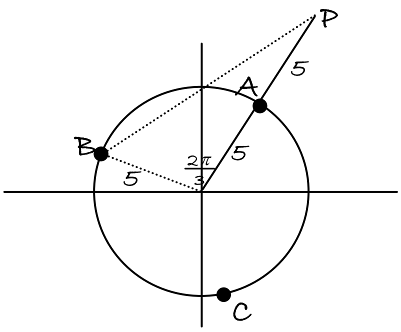
$a + bi - (b - a)(1 - i) = 2a - b + (2b - a)i$

$a + bi - i(b - a)(1 - i) = 2a - b + ai$

The sum of the vertices $= b + b + 2a - b + 2a - b + i(a + 2b - a + 2b - a + a)$

$= 4a + 4bi$

4. $|\alpha| = 5$ so circle is $|z| = 5$



(i) Since $|PA| = |OA|, |PA| = 5$

Using the cosine rule, $|PB|^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos \frac{2\pi}{3}$

$= 25 + 100 - 100 \times -\frac{1}{2}$

$= 175$

By symmetry $|PC| = |PB|$

so $PA \times PB \times PC = 5 \times \sqrt{175} \times \sqrt{175} = 875$

(ii) The point B can be expressed as $\alpha\omega$ where ω is a cube root of 1.

$$\begin{aligned}
PB^2 &= |2\alpha - \alpha\omega|^2 = |\alpha|^2 |2-\omega|^2 \\
&= 25(2-\omega)(2-\omega)^* \\
&= 25(2-\omega)(2-\omega^*) \\
&= 25(4 - 2\omega - 2\omega^* + \omega\omega^*) \\
&= 25(4 - 2\omega - 2\omega^* + 1) \\
&= 25(7 - 2(1 + \omega + \omega^*))
\end{aligned}$$

Since $1 + \omega + \omega^* = 0$, $PB^2 = 25 \times 7 = 175$

so $PA \times PB \times PC = 5 \times \sqrt{175} \times \sqrt{175} = 875$