Section 1: de Moivre's theorem

Solutions to Exercise level 2

1. (i) $\left|\sqrt{3}+i\right| = \sqrt{3+1} = 2$

$\arg(\sqrt{3}+i) = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$\begin{aligned}\left(\sqrt{3}+i\right)^{10} &= 2^{10}\left(\cos\tfrac{10\pi}{6} + i\sin\tfrac{10\pi}{6}\right)\\ &= 2^{10}\left(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}\right)\\ &= 2^{10}\left(\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3}\right)\\ &= 512 - 512i\sqrt{3}\end{aligned}$$

(ii) $|z_1| = \left(\sqrt{3+16}\right)^5 = 1573.6$

$|z_2| = \left(\sqrt{4+25}\right)^3 = 156.2$

$|z_3| = \left(\sqrt{5+9}\right)^4 = 196$

so z_1 has the largest modulus

(iii) $\arg z_1 = 5\times\left(-\arctan\frac{4}{\sqrt{3}}\right) = -5.81$

$\arg z_2 = 3\times\left(-\arctan\frac{5}{2}\right) = -3.57$

$\arg z_3 = 4\times\left(-\arctan\frac{3}{\sqrt{5}}\right) = -3.72$

Principal argument of $z_1 = -5.81 + 2\pi = 0.47$

Principal argument of $z_2 = 2\pi - 3.57 = 2.71$

Principal argument of $z_3 = 2\pi - 3.72 = 2.56$

so z_2 has the greatest principal argument.

2. (i) $|2-2i| = \sqrt{2^2+2^2} = 2\sqrt{2}$

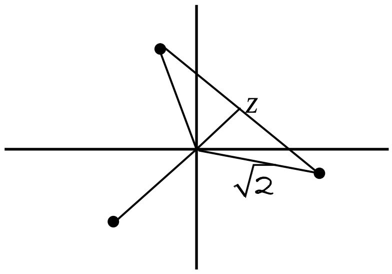
$2 - 2i$ is in the 4th quadrant, so $\arg(2-2i) = \arctan(-1) = -\frac{\pi}{4}$

(ii) Modulus $= \sqrt[3]{2\sqrt{2}} = \sqrt{2}$

Arguments are $\frac{1}{3}\times -\frac{\pi}{4} = -\frac{\pi}{12}$

$\frac{1}{3}\times\left(2\pi - \frac{\pi}{4}\right) = \frac{7\pi}{12}$

$\frac{1}{3}\times\left(-2\pi - \frac{\pi}{4}\right) = -\frac{9\pi}{12} = -\frac{3\pi}{4}$



(iii) Modulus $= \sqrt{2}\cos 60° = \frac{1}{2}\sqrt{2}$

The arguments can be found by adding or subtracting π from the arguments of the vertices of the triangle.

Arguments are $\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{4}$

(iv) $w = z^3$, so $|w| = |z|^3 = \frac{1}{8} \times 2\sqrt{2} = \frac{1}{4}\sqrt{2}$

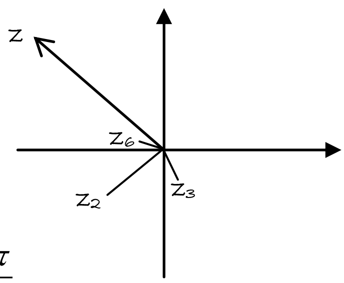
$\arg w = 3\arg z = 3 \times \frac{\pi}{4} = \frac{3\pi}{4}$

$w = \frac{1}{4}\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

$= \frac{1}{4}\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= -\frac{1}{4} + \frac{1}{4}i$

3. (i) $z = 64(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$



$\sqrt{z}$ has modulus 8 and argument $\dfrac{\pi}{3}$ or $-\dfrac{2\pi}{3}$

so $z_2 = 8(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))$

$\sqrt[3]{z}$ has modulus 4 and argument $\dfrac{2\pi}{9}$ or $\dfrac{8\pi}{9}$ or $-\dfrac{4\pi}{9}$

so $z_3 = 4(\cos(-\frac{4\pi}{9}) + i\sin(-\frac{4\pi}{9}))$

$\sqrt[6]{z}$ has modulus 2 and argument $\dfrac{\pi}{9}$ or $\dfrac{4\pi}{9}$ or $\dfrac{7\pi}{9}$ or …

so $z_6 = 2(\cos(\frac{7\pi}{9}) + i\sin(\frac{7\pi}{9}))$

(ii) $z_2 z_3 z_6$ has modulus $8 \times 4 \times 2 = 64$

$z_2 z_3 z_6$ has argument $-\dfrac{2\pi}{3} - \dfrac{4\pi}{9} + \dfrac{7\pi}{9} = -\dfrac{\pi}{3}$

$z_2 z_3 z_6 = 64(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$

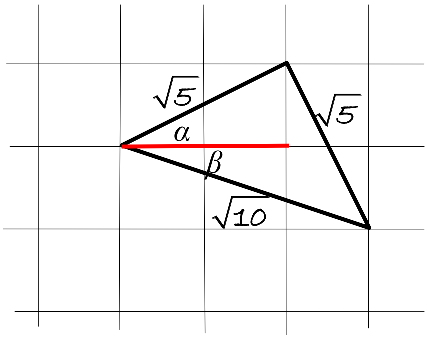
You might expect $z^{\frac{1}{2}} \times z^{\frac{1}{3}} \times z^{\frac{1}{6}} = z^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = z^1 = z$, but in this case it is not.

However, if you use the roots with the smallest arguments:

$$\frac{\pi}{3}+\frac{2\pi}{9}+\frac{\pi}{9}=\frac{2\pi}{3}$$

so in this case the product does give z.

4. (i)



From the side lengths, the triangle is a right-angled isosceles triangle, and so $\alpha+\beta=\frac{\pi}{4}$.

From the diagram, $\tan\alpha=\frac{1}{2}$ and $\tan\beta=\frac{1}{3}$,

so $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\frac{\pi}{4}$.

(ii) $z=2+i$

$|z|=\sqrt{2^2+1^2}=\sqrt{5}$

$w=\frac{3\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$

$|w|=\sqrt{\frac{9}{2}+\frac{1}{2}}=\sqrt{5}$

(iii) $\arg z=\arctan\frac{1}{2}$

$\arg w=\arctan\frac{1}{3}$

so from (i), $\arg z+\arg w=\frac{\pi}{4}$

(iv) $|m|=|zw|^8=\left(|z||w|\right)^8=5^8$

$\arg m=8\arg(zw)=8(\arg z+\arg w)=8\times\frac{\pi}{4}=2\pi$

so $m=5^8=390625$

z

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5. (i) Modulus of sixth roots are $\sqrt[6]{64} = 2$

Arguments of sixth roots are $\frac{\pi}{9} \pm \frac{n\pi}{3}$

so arguments are $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, -\frac{2\pi}{9}, -\frac{5\pi}{9}, -\frac{8\pi}{9}$

Sixth roots are $2(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9})$

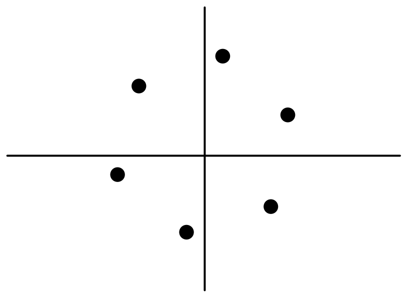
$2(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9})$

$2(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9})$

$2(\cos(-\frac{2\pi}{9}) + i\sin(-\frac{2\pi}{9}))$

$2(\cos(-\frac{5\pi}{9}) + i\sin(-\frac{5\pi}{9}))$

$2(\cos(-\frac{8\pi}{9}) + i\sin(-\frac{8\pi}{9}))$



(ii) The cube roots must form an equilateral triangle, so either the arguments are $\frac{\pi}{9}, \frac{7\pi}{9}, -\frac{5\pi}{9}$ or $\frac{4\pi}{9}, -\frac{2\pi}{9}, -\frac{8\pi}{9}$

So the argument of α could be $\frac{\pi}{3}$ or $-\frac{2\pi}{3}$

The modulus of α is $2^3 = 8$

So α could be $= 8(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

or $= 8(\cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}))$