Section 1: de Moivre's theorem

Solutions to Exercise level 1

1. (i) $(\cos 2\theta + i\sin 2\theta)^4 = \cos(4\times 2\theta) + i\sin(4\times 2\theta)$
$= \cos 8\theta + i\sin 8\theta$

(ii) $|1+\sqrt{3}i| = \sqrt{1+3} = 2$

$\arg(1+\sqrt{3}i) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$(1+\sqrt{3}i)^{12} = [2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})]^{12}$

$= 2^{12}(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})^{12}$

$= 2^{12}(\cos 4\pi + i\sin 4\pi)$

$= 2^{12}$ or 4096

(iii) $|1-i| = \sqrt{2}$

$\arg(1-i) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$1-i = \sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))$

$(1-i)^6 = [\sqrt{2}(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4}))]^6$

$= (\sqrt{2})^6(\cos(-\frac{6\pi}{4}) + i\sin(-\frac{6\pi}{4}))$

$= 2^3 \times i$

$= 8i$

(iv) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9 = \cos 3\pi + i\sin 3\pi$

$= -1$

2. (i) $|z_1| = 1^6 = 1$

$|z_2| = 1^4 = 1$

so they both have the same modulus

(ii) $\arg(z_1) = 6\times\frac{\pi}{4} = \frac{3}{2}\pi$ so principal argument is $-\frac{1}{2}\pi$

$\arg(z_2) = 4\times\frac{\pi}{6} = \frac{2}{3}\pi$ so principal argument is $\frac{2}{3}\pi$

so z_2 has the larger principal argument.

3. (i) $|w_1| = 1^{-4} = 1$

$|w_2| = 1^3 = 1$

so they both have the same modulus

(ii) $\arg(w_1) = -4 \times \frac{\pi}{3} = -\frac{4}{3}\pi$ so principal argument is $-\frac{4}{3}\pi$

$\arg(w_2) = 3 \times -\frac{\pi}{4} = -\frac{3}{4}\pi$ so principal argument is $-\frac{3}{4}\pi$

so w_2 has the larger principal argument

4. $\cos(\theta) + i\sin(\theta)$

$\cos(\theta) + i\sin(-\theta) = \cos(-\theta) + i\sin(-\theta)$

$\cos(-\theta) + i\sin(\theta) = \cos(\theta) + i\sin(\theta)$

$\cos(-\theta) + i\sin(-\theta)$

$\cos(\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta)$

$\cos(\theta) - i\sin(-\theta) = \cos(\theta) + i\sin(\theta)$

$\cos(-\theta) - i\sin(\theta) = \cos(-\theta) + i\sin(-\theta)$

$\cos(-\theta) - i\sin(-\theta) = \cos(\theta) + i\sin(\theta)$

Multiplying together gives modulus 1 and argument 0, so 1

Adding together gives $8\cos\theta$

5. $\left(1+\omega+2\omega^2\right)^9 = \left[(1+\omega+\omega^2)+\omega^2\right]^9$

$= \left(\omega^2\right)^9$ (since $1+\omega+\omega^2=0$ for a complex cube root of unity)

$= \omega^{18}$

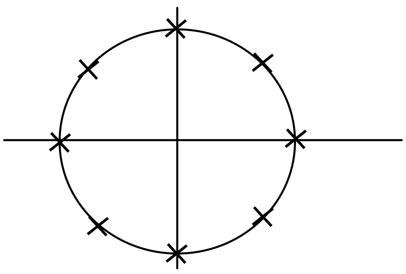
$= 1$ (since $\omega^3 = 1$)

6. $z^8 = 1$

$\therefore$ One root is $z = 1$

Hence all the 8 roots lie on a circle in the Argand diagram of radius 1.

The 8 roots are symmetrically placed on the circle with one root being $z = 1$.



(i) All the roots have modulus 1

The arguments of the roots are 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $-\frac{\pi}{4}$, $-\frac{\pi}{2}$, $-\frac{3\pi}{4}$.

(ii) $\cos 0 + i\sin 0 = 1$

$\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}(1+i)$
$\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$
$\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}(-1+i)$
$\cos\pi+i\sin\pi=-1$
$\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}(1-i)$
$\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=-i$
$\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}(-1-i)$

The roots are 1, $\frac{1}{\sqrt{2}}(1+i)$, i, $\frac{1}{\sqrt{2}}(-1+i)$, -1, $\frac{1}{\sqrt{2}}(1-i)$, $-i$, $\frac{1}{\sqrt{2}}(-1-i)$.

7. If ω is a complex seventh root of unity, $\omega^7=1$ and $\omega\neq1$,
$(\omega^2)^7=\omega^{14}=1$
so ω^2 is also a complex seventh root of unity and $\omega\neq\omega^2$ (since $\omega\neq1$)
so ω^2 is a different complex seventh root of unity.

Similarly $\left(\omega^3\right)^7=\omega^{21}=1$
so ω^3 is also a complex seventh root of unity and again $\omega^3\neq\omega^2$ or ω

The seven roots of unity are ω, ω^2, ω^3, ω^4, ω^5, ω^6 and ω^7 (which is 1).