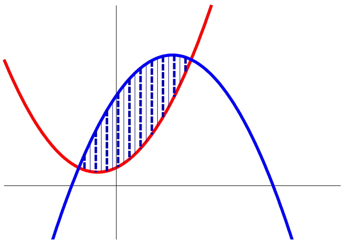
Topic assessment

1. The diagram below shows the graphs of $y = x^2 + x + 1$ and $y = 5 + 3x - x^2$.



(i) Find the coordinates of the points of intersection of the curves. [3]
(ii) Calculate the shaded area. [6]

2. Find the following indefinite integrals, using any appropriate method.

(i) $\int \frac{x^2}{(x^3 + 2)^2}\,\mathrm{d}x$ [4]

(ii) $\int \frac{\mathrm{e}^x}{1 + \mathrm{e}^x}\,\mathrm{d}x$ [4]

3. Evaluate

(i) $\int_0^2 x\mathrm{e}^{x^2}\,\mathrm{d}x$. [5]

(ii) $\int_0^{\pi/2} \frac{\cos x}{\sin x + 1}\,\mathrm{d}x$. [5]

(iii) $\int_0^{\pi/2} \sin^2 x \cos x\,\mathrm{d}x$. [5]

4. Evaluate $\int_0^1 x\sqrt{(1 - x)}\,\mathrm{d}x$. [5]

5. Evaluate $\int_1^{\mathrm{e}} \frac{1}{x^2} \ln x\,\mathrm{d}x$. [5]

6. Find $\int x \sin 3x\,\mathrm{d}x$. [4]

7. Express $\mathrm{f}(x) = \frac{x}{(x+1)(x+2)}$ in partial fractions and hence evaluate $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$ leaving your answer in logarithmic form. [6]

8. Using a suitable method integrate

(i) $\int \frac{x}{(x^2 - 1)^3}\,\mathrm{d}x$. [4]

(ii) $\int \frac{x}{x - 1}\,\mathrm{d}x$ [4]

Total 60 marks

Edexcel A level Maths Integration Assessment solns

Solutions to topic assessment

1. (i) At intersections, $x^2 + x + 1 = 5 + 3x - x^2$

$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1 \text{ or } x = 2$$

[3]

(ii) Area under $y = 5 + 3x - x^2$ between x = -1 and x = 2

$$= \int_{-1}^{2} (5 + 3x - x^2)\,dx$$
$$= \left[5x + \tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\right]_{-1}^{2}$$
$$= \left(10 + 6 - \tfrac{8}{3}\right) - \left(-5 + \tfrac{3}{2} + \tfrac{1}{3}\right)$$
$$= 16.5$$

Area under $y = x^2 + x + 1$ between x = -1 and x = 2

$$= \int_{-1}^{2} (x^2 + x + 1)\,dx$$
$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x\right]_{-1}^{2}$$
$$= \left(\tfrac{8}{3} + 2 + 2\right) - \left(-\tfrac{1}{3} + \tfrac{1}{2} - 1\right)$$
$$= 7.5$$

Area between graphs $= 16.5 - 7.5 = 9$ square units.

[6]

2. (i) The derivative of $x^3 + 2$ is $3x^2$.

By inspection, $\displaystyle\int \frac{x^2}{(x^3+2)^2}\,dx = \tfrac{1}{3}\int 3x^2 (x^3+2)^{-2}\,dx$

$$= \tfrac{1}{3} \times -1(x^3+2)^{-1} + c$$
$$= -\frac{1}{3(x^3+2)} + c$$

Alternatively, use the substitution $u = x^3 + 2$

[4]

(ii) The derivative of e^x is e^x.

By inspection, $\displaystyle\int \frac{e^x}{1+e^x}\,dx = \ln\left|1 + e^x\right| + c$

[4]

3. (i) The derivative of e^{x^2} is $2xe^{x^2}$.

By inspection, $\displaystyle\int_0^2 xe^{x^2}\,dx = \tfrac{1}{2}\int_0^2 2xe^{x^2}\,dx$

$$= \tfrac{1}{2}\left[e^{x^2}\right]_0^2$$
$$= \tfrac{1}{2}\left(e^4 - 1\right)$$

[5]

(ii) The derivative of $\sin x + 1$ is $\cos x$.

By inspection, $\int_0^{\pi/2} \frac{\cos x}{\sin x + 1}\, dx = [\ln(\sin x + 1)]_0^{\frac{\pi}{2}}$

$= \ln(\sin \frac{\pi}{2} + 1) - \ln(\sin 0 + 1)$

$= \ln 2$

[5]

(iii) The derivative of $\sin x$ is $\cos x$.

By inspection, $\int_0^{\pi/2} \sin^2 x \cos x\, dx = \left[\frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{3}\left(\sin^3 \frac{\pi}{3} - \sin^3 0\right)$

$= \frac{1}{3}$

[5]

4. $u = 1 - x \Rightarrow \frac{du}{dx} = -1 \Rightarrow dx = -du$

$x = 1 - u$

When $x = 0$, $u = 1$

When $x = 1$, $u = 0$

$\int_0^1 x\sqrt{(1-x)}\, dx = \int_1^0 (1-u)\sqrt{u} \times -du$

$= -\int_1^0 (u^{\frac{1}{2}} - u^{\frac{3}{2}})\, du$

$= -\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^0$

$= -0 + \left(\frac{2}{3} - \frac{2}{5}\right)$

$= \frac{4}{15}$

[5]

5. Let $u = \ln x \quad \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = \frac{1}{x^2} = x^{-2} \quad \Rightarrow v = -x^{-1} = -\frac{1}{x}$

Using integration by parts, $\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

$$\int_1^e \frac{1}{x^2}\ln x\,dx = \left[\ln x \times -\frac{1}{x}\right]_1^e - \int_1^e \left(\frac{1}{x}\times -\frac{1}{x}\right)dx$$
$$= \left[-\frac{1}{x}\ln x\right]_1^e + \int_1^e x^{-2}\,dx$$
$$= \left[-\frac{1}{x}\ln x - x^{-1}\right]_1^e$$
$$= \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^e$$
$$= \left(-\frac{1}{e}\ln e - \frac{1}{e}\right) - (-1\ln 1 - 1)$$
$$= -\frac{1}{e} - \frac{1}{e} + 1$$
$$= 1 - \frac{2}{e}$$

[5]

6. Let $u = x \quad \Rightarrow \dfrac{du}{dx} = 1$

Let $\dfrac{dv}{dx} = \sin 3x \quad \Rightarrow v = -\tfrac{1}{3}\cos 3x$.

Using integration by parts, $\int u\dfrac{dv}{dx}dx = uv - \int v\dfrac{du}{dx}dx$

$$\int x\sin 3x\,dx = x \times -\tfrac{1}{3}\cos 3x - \int 1 \times -\tfrac{1}{3}\cos 3x\,dx$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\int \cos 3x\,dx$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{3}\times\tfrac{1}{3}\sin 3x + c$$
$$= -\tfrac{1}{3}x\cos 3x + \tfrac{1}{9}\sin 3x + c$$

[4]

7. $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$

$x = A(x+2) + B(x+1)$

Putting $x = -2 \quad \Rightarrow -2 = -B \quad \Rightarrow B = 2$

Putting $x = -1 \quad \Rightarrow -1 = A \quad \Rightarrow A = -1$

$$\frac{x}{(x+1)(x+2)} = \frac{2}{x+2} - \frac{1}{x+1}$$

$$\int_0^2 f(x)\,dx = \int_0^2 \left(\frac{2}{x+2} - \frac{1}{x+1}\right)dx$$
$$= \left[2\ln(x+2) - \ln(x+1)\right]_0^2$$
$$= 2\ln 4 - \ln 3 - (2\ln 2 - \ln 1)$$
$$= \ln \frac{4^2}{3 \times 2^2}$$
$$= \ln \tfrac{4}{3}$$

[6]

8\. (i) The derivative of $x^2 - 1$ is $2x$, so this integral can be done by inspection. (Alternatively, use the substitution $u = x^2 - 1$).

$$\int \frac{x}{(x^2-1)^3}\,dx = \tfrac{1}{2}\int 2x(x^2-1)^{-3}\,dx$$
$$= \tfrac{1}{2} \times -\tfrac{1}{2}(x^2-1)^{-2} + c$$
$$= -\frac{1}{4(x^2-1)^2} + c$$

[4]

(ii) Let $u = x - 1 \quad \Rightarrow \dfrac{du}{dx} = 1$

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du$$
$$= \int \left(1 + \frac{1}{u}\right)du$$
$$= u + \ln u + c$$
$$= x - 1 + \ln(x-1) + c$$
$$= x + \ln(x-1) + k$$

Replacing $c - 1$ with a new constant k

[4]