Section 4: Finding distances

Solutions to Exercise level 2

1. (i) Direction vector of line is $\begin{pmatrix}3\\1\\-8\end{pmatrix}-\begin{pmatrix}6\\-5\\1\end{pmatrix}=\begin{pmatrix}-3\\6\\-9\end{pmatrix}=-3\begin{pmatrix}1\\-2\\3\end{pmatrix}$

An equation for the line is $\underset{\sim}{r}=\begin{pmatrix}6\\-5\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$.

(ii) The direction vector of the line is parallel to the vector $\begin{pmatrix}1\\-2\\3\end{pmatrix}$, which is the vector perpendicular to the plane, so the line is perpendicular to the plane.

(iii) $\left[\begin{pmatrix}6\\-5\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}\right].\begin{pmatrix}1\\-2\\3\end{pmatrix}=-9$

$\begin{pmatrix}6\\-5\\1\end{pmatrix}.\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}.\begin{pmatrix}1\\-2\\3\end{pmatrix}=-9$

$6+10+3+\lambda(1+4+9)=-9$

$19+14\lambda=-9$

$14\lambda=-28$

$\lambda=-2$

Point of intersection is $\underset{\sim}{r}=\begin{pmatrix}6\\-5\\1\end{pmatrix}-2\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}4\\-1\\-5\end{pmatrix}$

Coordinates of point of intersection are (4, -1, -5).

(iv) The shortest distance from a point to a plane is the perpendicular distance, so this is the distance between (6, -5, 1) and (4, -1, -5).

Distance $=\sqrt{(6-4)^2+(-5-(-1))^2+(1-(-5))^2}$

$=\sqrt{2^2+(-4)^2+6^2}$

$=\sqrt{56}$

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2\. (i) The shortest distance of a point (α, β, γ) from a plane

$n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$

The point is (1, 4, -2) and the plane is $2x - 4y + z - 3 = 0$

$$\text{Shortest distance } = \frac{|(2\times 1) + (-4\times 4) + (1\times -2) - 3|}{\sqrt{2^2 + (-4)^2 + 1^2}}$$
$$= \frac{|2 - 16 - 2 - 3|}{\sqrt{4 + 16 + 1}}$$
$$= \frac{19}{\sqrt{21}}$$

(ii) The shortest distance of a point (α, β, γ) from a plane

$n_1x + n_2y + n_3z + d = 0$ is $\dfrac{|n_1\alpha + n_2\beta + n_3\gamma + d|}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$

The point is (-3, 0, 1) and the plane is $-2x + 3y + z - 2 = 0$

$$\text{Shortest distance is } = \frac{|(-2\times -3) + (3\times 0) + (1\times 1) - 2|}{\sqrt{(-2)^2 + 3^2 + 1^2}}$$
$$= \frac{|6 + 0 + 1 - 2|}{\sqrt{4 + 9 + 1}}$$
$$= \frac{5}{\sqrt{14}}$$

3\. (i) Let the point on the line that is closest to P be M.

$$\overrightarrow{OM} = \begin{pmatrix} 1-2\lambda \\ 2+3\lambda \\ \lambda \end{pmatrix}$$

$$\overrightarrow{PM} = \begin{pmatrix} 1-2\lambda \\ 2+3\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3-2\lambda \\ 3+3\lambda \\ -2+\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ is perpendicular to the line, so $\begin{pmatrix} -3-2\lambda \\ 3+3\lambda \\ -2+\lambda \end{pmatrix} . \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = 0$

$$-2(-3-2\lambda) + 3(3+3\lambda) + (-2+\lambda) = 0$$
$$6 + 4\lambda + 9 + 9\lambda - 2 + \lambda = 0$$
$$14\lambda = -13$$
$$\lambda = -\frac{13}{14}$$

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$$\overrightarrow{PM} = \begin{pmatrix} -3+\frac{26}{14} \\ 3-\frac{39}{14} \\ -2-\frac{13}{14} \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -16 \\ 3 \\ -41 \end{pmatrix}$$

$$\left|\overrightarrow{PM}\right| = \frac{1}{14}\sqrt{16^2+3^2+41^2} = \frac{\sqrt{1946}}{14} = \sqrt{\frac{139}{14}}$$

$$\text{Distance of point from line } = \frac{\sqrt{139}}{\sqrt{14}} = 3.15 \quad (3 \text{ s.f.})$$

(ii) Let the point on the line that is closest to P be M.

$$\overrightarrow{OM} = \begin{pmatrix} 5+\lambda \\ -1+4\lambda \\ 3-\lambda \end{pmatrix}$$

$$\overrightarrow{PM} = \begin{pmatrix} 5+\lambda \\ -1+4\lambda \\ 3-\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ -1+4\lambda \\ 6-\lambda \end{pmatrix}$$

$\overrightarrow{PM}$ is perpendicular to the line, so $\begin{pmatrix} 3+\lambda \\ -1+4\lambda \\ 6-\lambda \end{pmatrix} . \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = 0$

$$3+\lambda+4(-1+4\lambda)-(6-\lambda)=0$$

$$3+\lambda-4+16\lambda-6+\lambda=0$$

$$18\lambda=7$$

$$\lambda=\frac{7}{18}$$

$$\overrightarrow{PM} = \begin{pmatrix} 3+\frac{7}{18} \\ -1+\frac{28}{18} \\ 6-\frac{7}{18} \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 61 \\ 10 \\ 101 \end{pmatrix}$$

$$\left|\overrightarrow{PM}\right| = \frac{1}{18}\sqrt{61^2+10^2+101^2} = \frac{1}{18}\sqrt{14022} = \sqrt{\frac{779}{18}}$$

$$\text{Distance of point from line } = \frac{\sqrt{779}}{\sqrt{18}} = 6.58 \quad (3 \text{ s.f.})$$

4. (i) $\underline{r} = \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$

(ii) This is equivalent to finding the distance between (1, 3, -6) and the line

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}.$$

Let Q be the point on the line nearest to P (1, 3, -6).

$$\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1-3\lambda \\ 5+\lambda \end{pmatrix}$$

$$\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1-3\lambda \\ 5+\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ -2-3\lambda \\ 11+\lambda \end{pmatrix}$$

$\overrightarrow{PQ}$ is perpendicular to the line, so $\begin{pmatrix} 1 \\ -2-3\lambda \\ 11+\lambda \end{pmatrix} . \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = 0$

$$-3(-2-3\lambda) + 11 + \lambda = 0$$
$$10\lambda = -17$$
$$\lambda = -1.7$$

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 3.1 \\ 9.3 \end{pmatrix}$$

$$\overrightarrow{PQ} = \sqrt{1^2 + 3.1^2 + 9.3^2} = 9.85 \text{ (3 s.f.)}$$

5. (i) If the lines intersect, there are values of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ for which

$$\begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

(1) $3 + \lambda = 2\mu$

(2) $5 = 1 + 5\mu$

(3) $-2 + 2\lambda = 3 - \mu$

(2) $\Rightarrow \mu = \frac{4}{5}$

Substituting into (1) $\Rightarrow 3 + \lambda = \frac{8}{5} \Rightarrow \lambda = -\frac{7}{5}$

Substituting into (3) $\Rightarrow -2 + 2\lambda = 3 - \frac{4}{5} \Rightarrow 2\lambda = \frac{21}{5} \Rightarrow \lambda = \frac{21}{10}$

Since the equations are not consistent, the lines do not intersect.

The direction vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ are not parallel, so the lines are not parallel. Therefore the lines are skew.

(ii) Let P and Q be the points on the two lines that are closest together.

So P has position vector $\underset{\sim}{r} = \begin{pmatrix} 3+\lambda \\ 5 \\ -2+2\lambda \end{pmatrix}$

and Q has position vector $\underset{\sim}{r} = \begin{pmatrix} 2\mu \\ 1+5\mu \\ 3-\mu \end{pmatrix}$

$$\overrightarrow{PQ} = \begin{pmatrix} 2\mu \\ 1+5\mu \\ 3-\mu \end{pmatrix} - \begin{pmatrix} 3+\lambda \\ 5 \\ -2+2\lambda \end{pmatrix} = \begin{pmatrix} -3-\lambda+2\mu \\ -4+5\mu \\ 5-2\lambda-\mu \end{pmatrix}$$

PQ is perpendicular to both lines

so $\begin{pmatrix} -3-\lambda+2\mu \\ -4+5\mu \\ 5-2\lambda-\mu \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$

$$-3-\lambda+2\mu+2(5-2\lambda-\mu) = 0$$

$$-5\lambda+7 = 0$$

$$\lambda = \tfrac{7}{5}$$

and $\begin{pmatrix} -3-\lambda+2\mu \\ -4+5\mu \\ 5-2\lambda-\mu \end{pmatrix} . \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} = 0$

$$2(-3-\lambda+2\mu)+5(-4+5\mu)-(5-2\lambda-\mu) = 0$$

$$30\mu-31 = 0$$

$$\mu = \tfrac{31}{30}$$

so $\overrightarrow{PQ} = \begin{pmatrix} -3-\frac{7}{5}+\frac{31}{15} \\ -4+\frac{31}{6} \\ 5-\frac{14}{5}-\frac{31}{30} \end{pmatrix} = \begin{pmatrix} -\frac{7}{3} \\ \frac{7}{6} \\ \frac{7}{6} \end{pmatrix}$

$$\left|\overrightarrow{PQ}\right| = \tfrac{1}{6}\sqrt{14^2+7^2+7^2} = \tfrac{1}{6}\sqrt{294} = \tfrac{7}{6}\sqrt{6}$$

6. The lines can be written as $\underset{\sim}{r} = \begin{pmatrix} 7 \\ 3 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} -8 \\ -1 \\ 3 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$

Let P and Q be the points on the two lines that are closest together.

So P has position vector $\underset{\sim}{r} = \begin{pmatrix} 7+3\lambda \\ 3+\lambda \\ 1-2\lambda \end{pmatrix}$

and Q has position vector $\underset{\sim}{r} = \begin{pmatrix} -8+3\mu \\ -1+2\mu \\ 3-\mu \end{pmatrix}$

$$\overrightarrow{PQ} = \begin{pmatrix} -8+3\mu \\ -1+2\mu \\ 3-\mu \end{pmatrix} - \begin{pmatrix} 7+3\lambda \\ 3+\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} -15-3\lambda+3\mu \\ -4-\lambda+2\mu \\ 2+2\lambda-\mu \end{pmatrix}$$

PQ is perpendicular to both lines

so $\begin{pmatrix} -15-3\lambda+3\mu \\ -4-\lambda+2\mu \\ 2+2\lambda-\mu \end{pmatrix} . \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = 0$

$$3(-15-3\lambda+3\mu)+(-4-\lambda+2\mu)-2(2+2\lambda-\mu)=0$$

$$-14\lambda+13\mu=53$$

and $\begin{pmatrix} -15-3\lambda+3\mu \\ -4-\lambda+2\mu \\ 2+2\lambda-\mu \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = 0$

$$3(-15-3\lambda+3\mu)+2(-4-\lambda+2\mu)-(2+2\lambda-\mu)=0$$

$$-13\lambda+14\mu=55$$

Solving these equations simultaneously gives $\lambda=-1, \mu=3$

so $\overrightarrow{PQ} = \begin{pmatrix} -15+3+9 \\ -4+1+6 \\ 2-2-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$\left|\overrightarrow{PQ}\right| = 3\sqrt{1^2+1^2+1^2} = 3\sqrt{3}$

P = (4, 2, 3) and Q = (1, 5, 0)