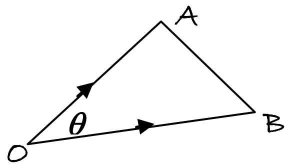
Section 1: The scalar product

Exercise level 3 solutions

1.



$\overrightarrow{OA} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} \quad \Rightarrow \left|\overrightarrow{OA}\right|^2 = a_1^2 + a_2^2$

$\overrightarrow{OB} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} \quad \Rightarrow \left|\overrightarrow{OB}\right|^2 = b_1^2 + b_2^2$

$\overrightarrow{AB} = (b_1 - a_1)\underset{\sim}{i} + (b_2 - a_2)\underset{\sim}{j} \quad \Rightarrow \left|\overrightarrow{AB}\right|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2$

Using the cosine rule:

$$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{OA}\right|^2 + \left|\overrightarrow{OB}\right|^2 - 2\left|\overrightarrow{OA}\right|\left|\overrightarrow{OB}\right|\cos\theta$$

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 = a_1^2 + a_2^2 + b_1^2 + b_2^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$-2a_1b_1 - 2a_2b_2 = -2|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

$$a_1b_1 + a_2b_2 = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$$

2. Let $\underset{\sim}{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} c \\ d \end{pmatrix}$

$\underset{\sim}{u}.\underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}|\cos\theta$

$\begin{pmatrix} a \\ b \end{pmatrix}.\begin{pmatrix} c \\ d \end{pmatrix} = \left|\begin{pmatrix} a \\ b \end{pmatrix}\right|\left|\begin{pmatrix} c \\ d \end{pmatrix}\right|\cos\theta$

$ac + bd = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}\cos\theta$

$(ac + bd)^2 = (a^2 + b^2)(c^2 + d^2)\cos^2\theta$

Since $0 \le \cos^2\theta \le 1$, $(ac + bd)^2 \le (a^2 + b^2)(c^2 + d^2)$

The equality holds when $\cos\theta = \pm 1$, which means that $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are parallel, so

$\begin{pmatrix} a \\ b \end{pmatrix} = k\begin{pmatrix} c \\ d \end{pmatrix}$ for some k

so $a = kc$ and $b = kd$

Dividing gives $\frac{a}{b} = \frac{c}{d}$ or $ad = bc$

3. Let $\underset{\sim}{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\underset{\sim}{v} = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a-c \\ b-d \end{pmatrix}$$

If perpendicular, $(\underset{\sim}{u} + \underset{\sim}{v}).(\underset{\sim}{u} - \underset{\sim}{v}) = 0$

$$\begin{pmatrix} a+c \\ b+d \end{pmatrix} . \begin{pmatrix} a-c \\ b-d \end{pmatrix} = 0$$

$$(a+c)(a-c) + (b+d)(b-d) = 0$$

$$a^2 - c^2 + b^2 - d^2 = 0$$

$$a^2 + b^2 = c^2 + d^2$$

$$|\underset{\sim}{u}|^2 = |\underset{\sim}{v}|^2$$

so $|\underset{\sim}{u}| = |\underset{\sim}{v}|$