Section 1: The scalar product

Solutions to Exercise level 2

1. (i) $\overrightarrow{AB} = \begin{pmatrix}0\\3\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}2\\5\end{pmatrix} - \begin{pmatrix}0\\3\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$

$\overrightarrow{CD} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}1\\-3\end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix}3\\2\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$

(ii) AB is parallel to DC, and BC is parallel to AD, ABCD is a parallelogram.

(iii) Angle ABC is angle between vectors $\begin{pmatrix}-1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$.

$\begin{pmatrix}-1\\3\end{pmatrix}.\begin{pmatrix}1\\1\end{pmatrix} = (-1\times 1) + (3\times 1) = 2$

$\left|\begin{pmatrix}-1\\3\end{pmatrix}\right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

$\left|\begin{pmatrix}1\\1\end{pmatrix}\right| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\cos\theta = \dfrac{2}{\sqrt{10}\sqrt{2}}$

$\theta = 63.4^\circ$

By symmetry, angle CDA is also equal to 63.4°, and angles BCD and DAB are both 180° - 63.4° = 116.6°.

2. (i) $\overrightarrow{AB} = \begin{pmatrix}4\\6\end{pmatrix} - \begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix}10\\3\end{pmatrix} - \begin{pmatrix}4\\6\end{pmatrix} = \begin{pmatrix}6\\-3\end{pmatrix}$

$\overrightarrow{AB}.\overrightarrow{BC} = \begin{pmatrix}2\\4\end{pmatrix}.\begin{pmatrix}6\\-3\end{pmatrix} = (2\times 6) + (4\times -3) = 12 - 12 = 0$

The lines AB and BC are perpendicular, therefore angle ABC is a right angle.

(ii) Let the midpoint of AC be M.

$$\overrightarrow{OM} = \overrightarrow{OA} + \tfrac{1}{2}\overrightarrow{AC}$$
$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \tfrac{1}{2}\left(\begin{pmatrix} 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2.5 \end{pmatrix}$$

(iii) If ABCD is a rectangle, $\overrightarrow{AD} = \overrightarrow{BC} = \begin{pmatrix} 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$$

The coordinates of D are (8, -1).

3. (i) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel, then there is a number c for which $\underset{\sim}{a} = k\underset{\sim}{b}$.

$$2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k} = c(3\underset{\sim}{i} + s\underset{\sim}{j} + t\underset{\sim}{k})$$

Comparing coefficients of $\underset{\sim}{i}$: $2 = 3c \quad \Rightarrow c = \tfrac{2}{3}$

Comparing coefficients of $\underset{\sim}{j}$: $6 = cs \quad \Rightarrow 6 = \tfrac{2}{3}s \quad \Rightarrow s = 9$

Comparing coefficients of $\underset{\sim}{k}$: $-3 = ct \quad \Rightarrow -3 = \tfrac{2}{3}t \quad \Rightarrow t = -\tfrac{9}{2}$

(ii) If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, then $\underset{\sim}{a}.\underset{\sim}{b} = 0$.

$$\left(2\underset{\sim}{i} + 6\underset{\sim}{j} - 3\underset{\sim}{k}\right).\left(3\underset{\sim}{i} + s\underset{\sim}{j} + t\underset{\sim}{k}\right) = 0$$
$$6 + 6s - 3t = 0$$
$$t = 2s + 2$$

4. $\overrightarrow{AB} = (8\underset{\sim}{i} + 7\underset{\sim}{j} - 9\underset{\sim}{k}) - (-\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) = 9\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}$

$\overrightarrow{BC} = (2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}) - (8\underset{\sim}{i} + 7\underset{\sim}{j} - 9\underset{\sim}{k}) = -6\underset{\sim}{i} - 10\underset{\sim}{j} + 8\underset{\sim}{k}$

$\overrightarrow{AC} = (2\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}) - (-\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}) = 3\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k}$

$\overrightarrow{AB}.\overrightarrow{BC} = (9\underset{\sim}{i} + 5\underset{\sim}{j} - 12\underset{\sim}{k}).(-6\underset{\sim}{i} - 10\underset{\sim}{j} + 8\underset{\sim}{k}) = -54 - 50 - 96 \neq 0$

$\overrightarrow{BC}.\overrightarrow{AC} = (-6\underset{\sim}{i} - 10\underset{\sim}{j} + 8\underset{\sim}{k}).(3\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{k}) = -18 + 50 - 32 = 0$

so vectors BC and AC are perpendicular, and therefore the triangle has a right angle at C.

$$\begin{aligned}
\text{Area of triangle} &= \tfrac{1}{2}|BC||AC| \\
&= \tfrac{1}{2}\sqrt{(-6)^2+(-10)^2+8^2}\sqrt{3^2+(-5)^2+(-4)^2} \\
&= \tfrac{1}{2}\sqrt{200}\sqrt{50} \\
&= \tfrac{1}{2}\times 10\sqrt{2}\times 5\sqrt{2} \\
&= 25\times 2 \\
&= 50 \text{ square units}
\end{aligned}$$